Rate of convergence to the semi-circle law for the Deformed Gaussian Unitary Ensemble*

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Abstract: It is shown that the Kolmogorov distance between the expected spectral distribution function of an $n\times n$ matrix from the Deformed Gaussian Ensemble and the distribution function of the semi-circle law is of order $O(n^{-\frac{2}{3}+\nu})$.

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1 Introduction and results

Let $\mathbf{W}$ be a Hermitian matrix of order n with complex entries $W_{lj} = X_{lj} + i\,Y_{lj}$, $1 \leqslant l \leqslant j \leqslant n$. Assume that random variables $\{X_{lj}, Y_{lj}\}_{1\leqslant l\leqslant j\leqslant n}$ are independent. Furthermore, let $\mathbf{E}\,X_{kj} = \mathbf{E}\,Y_{kj} = 0$, $1 \leqslant k \leqslant j \leqslant n$ and $\mathbf{E}\,X_{ll}^2 = \sigma^2$, $l = 1, 2, \dots, n$, $\mathbf{E}\,X_{kj}^2 = \mathbf{E}\,Y_{kj}^2 = \sigma^2/2$, $1 \leqslant k < j \leqslant n$.

This class of matrices is called Wigner ensemble. For a fixed $n \geqslant 1$, denote by $\lambda_1 \leqslant \ldots \leqslant \lambda_n$ the eigenvalues of the matrix $\frac{1}{\sqrt{n}}\mathbf{W}$ and define their spectral distribution

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function

$$F_n(x) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{I}_{\{\lambda_j \leqslant x\}},$$

where $\mathbf{I}_{\{B\}}$ denotes the indicator of event B. In 1955 Wigner proved, using the moment approach, that

$$\mathbf{E}\, F_n(x) \to G(x), \tag{1}$$

where $G(x)$ is the distribution function of the semi-circle law with the density

$$g(x) = G'(x) = \frac{1}{2\pi\sigma^2}\sqrt{(4\sigma^2 - x^2)_+}.$$

He showed this result assuming that W_{lj} are real random variables that odd moments are zero and all even moments are finite (see [19]). Since then, a number of authors have proved this limit theorem under weaker conditions on W_{lj}. In particular, it was proved by Pastur in [18] that (1) follows when Lindeberg's condition holds for entries of the matrix $\mathbf{W}$. For more detailed information we refer to the surveys [2], [17].

The rate of convergence of $\mathbf{E}\, F_n(x)$ to the Wigner distribution has been intensively studied as well. Let

$$\Delta_n := \sup_x \left|\mathbf{E}\, F_n(x) - G(x)\right|.$$

Assuming that $\sup_{1\leqslant l\leqslant j<\infty} \mathbf{E}\, |W_{lj}|^4 < \infty$ Bai ([1], [3]) proved that $\Delta_n = O(n^{-1/4})$ and that $\Delta_n = O(n^{-1/3})$. He showed the bound $\Delta_n = O(n^{-1/2})$ as well assuming that $\sup_{1\leqslant l\leqslant j<\infty} \mathbf{E}\, |W_{lj}|^8 < \infty$ (see [4]). The bound $\Delta_n = O(n^{-1/2})$ assuming uniformly bounded fourth moments was proved independently by Götze and Tikhomirov in [11] and by Girko (see [7], [8]). In [11] it is shown that the same rate applies to convergence in probability.

In the Wigner ensemble, the class of the Gaussian Unitary Ensemble (GUE), that is of matrices where the real and imaginary parts of the entries have a Gaussian distribution, has been intensively studied for many years. The local limit distributions of their spacings are conjectured to be universal. In particular, an asymptotic expansion for the density of the expected spectral distribution function has been obtained by Deift et al., see [5], and Ercolani and McLaughlin, see [6] by Riemann-Hilbert Theory.

The Gaussian Unitary Ensemble plays a significant role in investigations of convergence rates as well. It has been shown by Götze and Tikhomirov in [13], by differential equation methods, that the convergence rate of expected spectral distribution function to the semi-circle law is of order $O(n^{-1})$ *uniformly* on the limiting spectrum.

Our goal in this paper is to investigate the convergence rate of the expected spectral distribution function of the matrices from a so-called Deformed Gaussian Unitary Ensemble (DGUE). This ensemble was introduced first by K. Johansson (see [14]) in the connection with the problem of universality of the local spacing distribution. This class of matrices is a subclass of the Wigner Ensemble and admits a joint density with respect to Lebesgue measure. In this class one can considerably improve the error bounds for the Wigner Ensemble.

Let $\mathbf{M} = (M_{lj})_{l,j=1}^n$ be a matrix from the DGU ensemble, that is

$$\mathbf{M} = \mathbf{W} + a\mathbf{H}, \quad a > 0,$$

where $\mathbf{W} = (W_{lj})_{l,j=1}^n$ is a Wigner matrix and $\mathbf{H} = (H_{lj})_{l,j=1}^n$ denotes an *independent* matrix from the GUE ensemble. Throughout this paper we shall assume that $\mathbf{E}\,|W_{lj}|^2 = \frac{1}{4}$, $1 \leqslant l \leqslant j \leqslant n$, and $\mathbf{E}\,|H_{lj}|^2 = 1$, $1 \leqslant l \leqslant j \leqslant n$. Furthermore, we shall assume that for any $k \geqslant 1$ $\sup_{1\leqslant l\leqslant j\leqslant n} \mathbf{E}\,|W_{lj}|^k < \infty$.

Let $F_n^a(x)$ be the spectral distribution function of the matrix $\frac{1}{\sqrt{n}}\mathbf{M}$ and $p_n^a(x)$ be the density of the expected spectral distribution function $\mathbf{E}\,F_n^a(x)$. Denote by $G^a(x)$ the distribution function of the semi-circle law with the density

$$g^a(x) = \frac{2}{\pi(1+4a^2)}\sqrt{(1+4a^2-x^2)_+}.$$

Suppose $y_1 \leqslant \ldots \leqslant y_n$ are the eigenvalues of the matrix $\frac{1}{\sqrt{n}}\mathbf{W}$; then by $p_n^a(x;y)$ we denote the density of the conditionally expected distribution function $\mathbf{E}\,(F_n^a(x)|y)$ for a given $n \times n$ matrix $\mathbf{W}$.

Our main results are the following

Theorem 1.1. Let $\nu > 0$, $c > 0$. There exists a positive constant $C(a,\nu,c)$ such that for any $x \in [-\sqrt{1+4a^2} + c\,n^{-\frac{1}{3}+\nu}, \sqrt{1+4a^2} - c\,n^{-\frac{1}{3}+\nu}]$ it follows that

$$|p_n^a(x) - g^a(x)| \leqslant \frac{C(a,\nu,c)}{n(1+4a^2-x^2)^2} + \mathbf{E}\,\left(p_n^a(x;y)\mathbf{I}_{\{y\notin\Omega_0\}}\right),$$

where $\mathbf{P}\,(\Omega_0) \geqslant 1 - \frac{1}{n}$.

Theorem 1.2. For any $\nu > 0$ there exists some constant $C(a,\nu) > 0$ such that

$$\sup_x |\mathbf{E}\,F_n^a(x) - G^a(x)| \leqslant C(a,\nu)\,n^{-\frac{2}{3}+\nu}.$$

2 The method of steepest descent

In the following we shall use Johansson's approach to study the eigenvalue distribution of DGUE. It follows from [14], (2.20) that, for $|u| \leqslant \sqrt{1+4a^2}$,

$$p_n^a(u;y) = n\int_\gamma \frac{dz}{2\pi}\int_\Gamma \frac{dw}{2\pi} g_n(z,w)\exp\Big\{n\big(f_n(w)-f_n(z)\big)\Big\},$$

$$g_n(z,w) = \frac{1}{a^4}\bigg(w + z - u - \frac{a^2}{n}\sum_{j=1}^n \frac{y_j}{(w-y_j)(z-y_j)}\bigg),$$

$$f_n(z) = \frac{1}{2a^2}(z^2 - 2uz) + \frac{1}{n}\sum_{j=1}^n \log(z-y_j),$$

where the contours γ and Γ will be defined below. Note that

$$g_n(z,w) = \frac{1}{a^2} f_n'(w) + \frac{z}{a^2}\frac{f_n'(z) - f_n'(w)}{z-w},$$
$$f_n'(z) = \frac{z-u}{a^2} - \frac{1}{n}\operatorname{Tr}\mathbf{R}(z),$$

where $\mathbf{R}(z) = (\frac{1}{\sqrt{n}}\mathbf{W} - z\mathbf{I})^{-1}$.
Let $f(z)$ denotes the limit of $f_n(z)$, then

$$f(z) = \frac{1}{2a^2}(z^2 - 2uz) + \frac{2}{\pi}\int_{-1}^{1} \log(z-t)\sqrt{1-t^2}dt$$
$$= \frac{z^2-2uz}{2a^2} + z^2 + \log(z+\sqrt{z^2-1}) - z\sqrt{z^2-1}.$$

The critical points ($f'(z_c) = 0$) for $f(z)$ are

$$z_c^{\pm} = u\,\frac{1+2a^2}{1+4a^2} \pm i\,p(u)\frac{2a^2}{1+4a^2},$$

where $p(u) = \sqrt{1+4a^2-u^2}$.
Note that

$$\left|z_c^{\pm}\right| = \sqrt{\frac{u^2+4a^4}{1+4a^2}}.$$

Now we consider the following representation

$$p_n^a(u;y) = n\int_\gamma \frac{dz}{2\pi}\int_\Gamma \frac{dw}{2\pi} g_n(z,w)\exp\Big\{n\big(f(w) - f(z_c^+)\big) - n\big(f(z) - f(z_c^+)\big) + n\big(\Delta_n(w) - \Delta_n(z)\big)\Big\}, \qquad (2)$$

where $\Delta_n(z) = f_n(z) - f(z)$.
Define transforms

$$S(w) = \frac{1}{2}(w + w^{-1}), \qquad S^{-1}(z) = z + \sqrt{z^2-1}.$$

Set

$$\cos\theta_c = \frac{u}{\sqrt{1+4a^2}}, \quad \theta_c \in [0,\pi], \qquad w_c^{\pm} = \sqrt{1+4a^2}\exp\{\pm\theta_c\} = S^{-1}(z_c^{\pm}).$$

Following the paper [14], we consider some contours (see Figure 1). Put, for some $\varepsilon > 0$ and $\delta > 0$,

$$\gamma_1^+(t) = S(\sqrt{1+4a^2}\,e^{i\,\delta} - t), \quad -\infty < t \leqslant 0,$$
$$\gamma_2^+(t) = S(\sqrt{1+4a^2}\,e^{i\,t}), \quad \delta \leqslant t \leqslant \theta_c - \varepsilon,$$
$$\gamma_3^+(t) = S(\sqrt{1+4a^2}\,e^{i\,t}), \quad \theta_c - \varepsilon \leqslant t \leqslant \theta_c + \varepsilon,$$
$$\gamma_4^+(t) = S(\sqrt{1+4a^2}\,e^{i\,t}), \quad \theta_c + \varepsilon \leqslant t \leqslant \pi - \delta,$$
$$\gamma_5^+(t) = S(\sqrt{1+4a^2}\,e^{i\,(\pi-\delta)} - t), \quad 0 \leqslant t < \infty.$$

Also, put $\gamma_\nu^-(t) = \overline{\gamma_\nu^+(t)}$, $1 \leqslant \nu \leqslant 5$ and set

$$\gamma = \sum_{\nu=1}^{5} (\gamma_\nu^+ - \gamma_\nu^-) = \gamma^+ - \gamma^-.$$

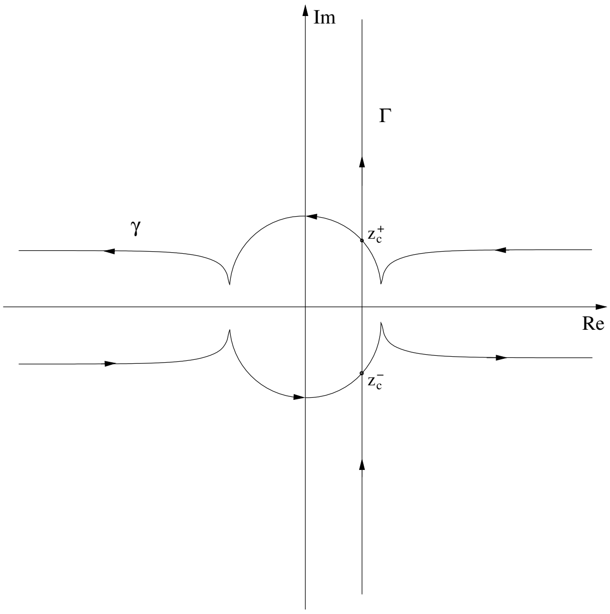

Fig. 1 Γ, γ.

Analogously we define the contour Γ. Set

$$\begin{aligned}
\Gamma_1^+(t) &= \operatorname{Re} z_c^+ + i\,t, \quad 0 \leqslant t \leqslant \operatorname{Im} z_c^+ - \varepsilon,\\
\Gamma_2^+(t) &= \operatorname{Re} z_c^+ + i\,t, \quad \operatorname{Im} z_c^+ + \varepsilon \leqslant t \leqslant \infty,\\
\Gamma_3^+(t) &= \operatorname{Re} z_c^+ + i\,t, \quad \operatorname{Im} z_c^+ - \varepsilon \leqslant t \leqslant \operatorname{Im} z_c^+ + \varepsilon.
\end{aligned}$$

Furthermore, let $\Gamma_\nu^-(t) = \overline{\Gamma_\nu^+(t)}$, $1 \leqslant \nu \leqslant 3$, and

$$\Gamma = \sum_{\nu=1}^{3} (\Gamma_\nu^+ - \Gamma_\nu^-) = \Gamma^+ - \Gamma^-.$$

Now we can take defined contours γ and Γ in (2).

Lemma 2.1. For $|\theta - \theta_c| \geqslant \frac{p(u)}{4\sqrt{1+4a^2}}$, $0 < \theta < \pi$ we have

$$\operatorname{Re}\Bigl(f\bigl(S(\sqrt{1+4a^2}\, e^{i\,\theta})\bigr) - f(z_c^+)\Bigr) \geqslant \frac{5(1+2a^2)}{1024a^2} \frac{p^4(u)}{(1+4a^2)^2}.$$

Proof. Consider the function $q(\theta) = \operatorname{Re} f\bigl(S(\sqrt{1+4a^2}\, e^{i\,\theta})\bigr)$, $0 < \theta < \pi$. Note that

$$\frac{d\,q(\theta)}{d\,\theta} = \frac{1+2a^2}{a^2} \sin\theta(\cos\theta_c - \cos\theta).$$

From here it follows, for $|\theta-\theta_c| \leqslant \frac{p(u)}{4\sqrt{1+4a^2}}$, that

$$\begin{aligned}\frac{d^2 q}{d\,\theta^2} &= \frac{1+2a^2}{a^2}\big((\cos\theta_c-\cos\theta)(\cos\theta_c+2\cos\theta)+\sin^2\theta_c\big)\\ &\geqslant \frac{1+2a^2}{a^2}\big(-3\,|\cos\theta_c-\cos\theta|+\sin^2\theta_c\big)\\ &\geqslant \frac{5(1+2a^2)}{32a^2}\,\frac{p^2(u)}{1+4a^2}.\end{aligned}$$

Note that if $|\theta-\theta_c| \geqslant \frac{p(u)}{4\sqrt{1+4a^2}}$ then

$$q(\theta)-q(\theta_c) \geqslant q(\theta')-q(\theta_c),$$

where $\theta' = \theta_c \pm \frac{p(u)}{4\sqrt{1+4a^2}}$. From the two last inequalities it follows that

$$q(\theta)-q(\theta_c) \geqslant \frac{(\theta'-\theta_c)^2}{2}\,\frac{5(1+2a^2)}{32a^2}\,\frac{p^2(u)}{1+4a^2} \geqslant \frac{5(1+2a^2)}{1024a^2}\,\frac{p^4(u)}{(1+4a^2)^2}.$$

□

Lemma 2.2. Let $\delta = \frac{a}{2\sqrt{1+4a^2}}\sin\theta_c$. Then we have

$$\operatorname{Re}\Big(f\big(S(\omega_\delta-t)\big)-f(z_c^+)\Big) \geqslant \frac{5(1+2a^2)}{1024a^2}\,\frac{p^4(u)}{(1+4a^2)^2}, \quad -\infty<t\leqslant 0, \tag{3}$$

$$\operatorname{Re}\Big(f\big(S(\omega_{\pi-\delta}-t)\big)-f(z_c^+)\Big) \geqslant \frac{5(1+2a^2)}{1024a^2}\,\frac{p^4(u)}{(1+4a^2)^2}, \quad 0\leqslant t<\infty, \tag{4}$$

where $\omega_\delta = \sqrt{1+4a^2}\,e^{i\,\delta}$, $\omega_{\pi-\delta} = \sqrt{1+4a^2}\,e^{i\,(\pi-\delta)}$.

Proof. Consider the function $k(t) = \operatorname{Re} f\big(S(\omega_\delta-t)\big)$, $-\infty<t\leqslant 0$.
Set $\omega_\delta - t = s(t)\,e^{i\,\theta(t)}$. After a simple calculation we get

$$\begin{aligned}\frac{d\,k(t)}{d\,t} = -\frac{1}{4a^2}\bigg\{&\bigg[\Big(s(t)+\frac{1+4a^2}{s(t)}\Big)\cos\theta(t)-2u\bigg]\\ &\times\bigg[1-\frac{1}{s^2(t)}\cos 2\theta(t)\bigg]-\frac{1}{s^2(t)}\sin 2\theta(t)\Big(s(t)-\frac{1+4a^2}{s(t)}\Big)\sin\theta(t)\bigg\}.\end{aligned}$$

The last equality implies

$$\begin{aligned}\frac{d\,k(t)}{d\,t} &\leqslant -\frac{1}{4a^2}\Big(1-\frac{1}{s^2(t)}\Big)\bigg[\Big(s(t)+\frac{1+4a^2}{s(t)}\Big)\cos\theta(t)\\ &\qquad -2\sqrt{1+4a^2}\Big(1+2\frac{(1+4a^2)\sin^2\delta}{s^2(t)\,(s^2(t)-1)}\Big)\cos\theta_c\bigg]\\ &\leqslant -\frac{\sqrt{1+4a^2}}{4a^2}\Big(1-\frac{1}{s^2(t)}\Big)\bigg[\Big(\frac{s(t)}{\sqrt{1+4a^2}}+\frac{\sqrt{1+4a^2}}{s(t)}\Big)\cos\theta(t)\\ &\qquad -2\Big(1+\frac{(1+4a^2)\sin^2\delta}{2a^2}\Big)\cos\theta_c\bigg]\end{aligned}$$

$$\leqslant -\frac{2}{\sqrt{1+4a^2}}\left[\cos\theta(t) - \left(1+\frac{1+4a^2}{2a^2}\sin^2\delta\right)\cos\theta_c\right]$$

$$\leqslant -\frac{1}{\sqrt{1+4a^2}}\left[\sin^2\theta_c - \delta^2\,\frac{1+5a^2}{a^2}\right].$$

Now, if we set in the last inequality

$$\delta = \frac{a}{2\sqrt{1+4a^2}}\sin\theta_c,$$

we get

$$\frac{d\,k(t)}{d\,t} \leqslant -\frac{p^2(u)}{2(1+4a^2)^{\frac{3}{2}}},\ -\infty < t \leqslant 0.$$

From here and Lemma 2.1 it follows that

$$\mathrm{Re}\left(f\big(S(\omega_\delta - t)\big) - f(z_c^+)\right) \geqslant k(0) - \mathrm{Re}\,f(z_c^+) \geqslant \frac{5(1+2a^2)}{1024a^2}\,\frac{p^4(u)}{(1+4a^2)^2}.$$

Thus, (3) is proved. The similar arguments can be used to prove (4). □

In what follows we shall assume that $\delta = \min\left\{\frac{\pi}{6},\ \frac{a}{2\sqrt{1+4a^2}}\sin\theta_c\right\}$.

Lemma 2.3. We have

$$\mathrm{Re}\left(f\big(S(\omega_\delta - t)\big) - f(z_c^+)\right) \geqslant \frac{\cos 2\delta}{16\,a^2}\,t^2, \quad t \leqslant -10\sqrt{1+4a^2},$$

$$\mathrm{Re}\left(f\big(S(\omega_{\pi-\delta} - t)\big) - f(z_c^+)\right) \geqslant \frac{\cos 2\delta}{16\,a^2}\,t^2, \quad t \geqslant 10\sqrt{1+4a^2},$$

where $\omega_\delta = \sqrt{1+4a^2}\,e^{i\,\delta}$, $\omega_{\pi-\delta} = \sqrt{1+4a^2}\,e^{i\,(\pi-\delta)}$.

Proof. It will be proved only the first inequality. The proof of the second one is analogous.

Let $k(t) = \mathrm{Re}\,f\big(S(\omega_\delta - t)\big)$, $\omega_\delta - t = s(t)\,e^{i\,\theta(t)}$, $-\infty < t \leqslant 0$. It is easy to check that

$$k(t) - k(0) \geqslant \frac{1}{16\,a^2}\,t^2\cos 2\theta(t) + \frac{1}{8\,a^2}\left(-1 - 2\sqrt{1+4a^2}\,t\cos\delta\right)\cos 2\theta(t)$$
$$+\frac{1}{16\,a^2}\left(t^2\cos 2\theta(t) - 8\,u\,s(t)\cos\theta(t)\right)$$

$$\geqslant \frac{1}{16\,a^2}\,t^2\cos 2\delta + \frac{1}{8\,a^2}\left(-1 - 2\sqrt{1+4a^2}\,t\cos\delta\right)\cos 2\theta(t)$$
$$+\frac{1}{16\,a^2}\left(t^2 + 8\sqrt{1+4a^2}\,t - 8(1+4a^2)\right)\cos 2\delta.$$

Since the two last items are positive for $t \leqslant -10\sqrt{1+4a^2}$, we have

$$\mathrm{Re}\left(f\big(S(\omega_\delta - t)\big) - f(z_c^+)\right) \geqslant q(t) - q(0) \geqslant \frac{\cos 2\delta}{16\,a^2}t^2,$$

for $t \leqslant -10\sqrt{1+4a^2}$. □

Let $\varkappa = \frac{1+2a^2}{1+4a^2}$, $t_c^{\pm} = \pm\frac{2a^2\, p(u)}{1+4a^2}$.

Lemma 2.4. The function $r(t) = \operatorname{Re} f(u\varkappa + i\,t)$ is increasing on $(-\infty, t_c^-) \cup (0, t_c^+)$ and decreasing on $(t_c^-, 0) \cup (t_c^+, +\infty)$.

Proof. It can be shown that

$$r'(t) = \left(\frac{2\sqrt{2}\,|u|\,\varkappa}{\sqrt{\sqrt{B}+A}} - \frac{1+2a^2}{a^2}\right)t,$$

where $A = u^2\varkappa^2 - t^2 - 1$, $B = A^2 + 4u^2\varkappa^2 t^2$. From the last expression it follows that $t = 0$ is a critical point of the function $r(t)$. Consider now the equation

$$\sqrt{\sqrt{B}+A} = 2\sqrt{2}\frac{a^2}{1+2a^2}\,|u|\,\varkappa.$$

Twice squaring we get

$$t^2\alpha = \frac{4a^4}{(1+2a^2)^2}\Big(1 - \alpha\, u^2\varkappa^2\Big),$$

where $\alpha = 1 - \frac{4a^4}{(1+2a^2)^2}$. Since $\frac{1}{\varkappa^2\alpha} = 1 + 4a^2$, we obtain

$$t^2 = \frac{4a^4}{(1+4a^2)^2}p^2(u).$$

Thus, the Lemma is proved. □

Lemma 2.5. For $|t - t_c^+| \leqslant \frac{a^6}{16(1+4a^2)^2}\frac{p(u)}{1+4a^2}$ we have

$$r''(t) \leqslant -\frac{1}{2a^2(1+4a^2)}p^2(u),$$

where $r(t)$ is the same as in Lemma 2.4.

Proof. Let $z = u\varkappa + i\,t$, $|t - t_c^+| \leqslant \mu\frac{p(u)}{1+4a^2}$, for some $\mu > 0$. Then it is easy to see that

$$\begin{aligned}\left|z^2 - z_c^{+2}\right| &= \left|(t_c^+ - t)(t_c^+ + t - i\,2u\varkappa)\right| \leqslant \left|t_c^+ - t\right|\left(2t_c^+ + \left|t_c^+ - t\right| + 2\,|u|\,\varkappa\right) \\ &\leqslant \frac{p(u)}{(1+4a^2)^{\frac{3}{2}}}\left(\mu^2 + 2(1+4a^2)\mu\right),\end{aligned}$$

and

$$\left|z_c^{+2} - 1\right| = \frac{1}{1+4a^2}\Big((1+2a^2)^2 - u^2\Big) \geqslant \frac{4a^4}{1+4a^2}.$$

From these inequalities it follows immediately, for $\mu \leqslant \frac{a^4}{1+4a^2}$, that

$$\left|z^2 - 1\right| \geqslant \left|z_c^{+2} - 1\right| - \left|z^2 - z_c^{+2}\right| \geqslant \frac{4a^4 - \mu^2 - 2(1+4a^2)\mu}{1+4a^2} \geqslant \frac{a^4}{1+4a^2}. \tag{5}$$

Using obtained above inequalities we get, for $\mu \leqslant \frac{a^4}{1+4a^2}$,

$$\left|\operatorname{Im}\sqrt{z^2-1}\right| \leqslant \left|\operatorname{Im}\sqrt{z_c^{+2}-1}\right| + \frac{\left|z^2-z_c^{+2}\right|}{\left|\sqrt{z^2-1}+\sqrt{z_c^{+2}-1}\right|}$$

$$\leqslant \varkappa\, p(u) + \frac{\left|z^2-z_c^{+2}\right|}{\left|\sqrt{z_c^{+2}-1}\right|} \leqslant p(u). \tag{6}$$

Now we consider the function $s(z) = -2z + 2\sqrt{z^2-1}$. It is the Stieltjes transform of the semi-circle law. A computation shows that

$$|\operatorname{Im} s''(z)| = \left|\operatorname{Im}\frac{2}{(\sqrt{z^2-1})^3}\right| = \frac{2}{(|z^2-1|)^3}\left|\operatorname{Im}(\sqrt{z^2-1})^3\right|$$

$$\leqslant \frac{8}{|z^2-1|^2}\left|\operatorname{Im}\sqrt{z^2-1}\right| \leqslant 8\,\frac{(1+4a^2)^2}{a^8}\,p(u). \tag{7}$$

The last inequality follows from (5) and (6). One can to check that

$$r''(t) - r''(t_c^+) = \operatorname{Re}\left\{s'\big(u\varkappa + i\,t\big) - s'\big(u\varkappa + i\,t_c^+\big)\right\}$$

$$\leqslant \left|t - t_c^+\right| \inf_{\left|t'-t_c^+\right| \leqslant \mu\frac{p(u)}{1+4a^2}} \left|\operatorname{Im} s''\big(u\varkappa + i\,t'\big)\right|.$$

From here and (7) we obtain

$$r''(t) - r''(t_c^+) \leqslant \mu\,\frac{8(1+4a^2)}{a^8} p^2(u).$$

Noting that

$$r''(t_c^+) \leqslant -\frac{1}{a^2(1+2a^2)}\,p^2(u),$$

we get

$$r''(t) \leqslant \left(\mu\,\frac{8(1+4a^2)}{a^8} - \frac{1}{a^2(1+2a^2)}\right) p^2(u) \leqslant -\frac{1}{2a^2\,(1+4a^2)} p^2(u),$$

for $\mu \leqslant \frac{a^6}{16(1+4a^2)^2}$.
The Lemma is proved. □

Lemma 2.6. Let $r(t) = \operatorname{Re} f(u\varkappa + i\,t)$. Then for $|t - t_c^+| \geqslant \frac{a^6}{16(1+4a^2)^2}\frac{p(u)}{1+4a^2}$, $0 \leqslant t < +\infty$ we have

$$r(t) - r(t_c^+) \leqslant -\frac{a^{10}}{1024}\,\frac{p^4(u)}{(1+4a^2)^7}.$$

Proof. By Lemma 2.4 we have, for $|t - t_c^+| \geqslant \frac{a^6}{16(1+4a^2)^2}\frac{p(u)}{1+4a^2}$, $0 \leqslant t < +\infty$,

$$r(t) - r(t_c^+) \leqslant r(t') - r(t_c^+),$$

where $t' = t_c^+ \pm \frac{a^6}{16(1+4a^2)^2}\frac{p(u)}{1+4a^2}$.
Applying Lemma 2.5 yields

$$r(t) - r(t_c^+) \leqslant -\frac{p^2(u)}{4a^2(1+4a^2)}(t' - t_c^+)^2 \leqslant -\frac{a^{10}}{1024}\,\frac{p^4(u)}{(1+4a^2)^7}.$$

The Lemma follows. □

Lemma 2.7. Let $r(t) = \mathrm{Re}\, f(u\varkappa + i\,t)$. Then for all $t \geqslant 2\sqrt{2}a$ we have

$$r(t) - r(t_c^+) \leqslant -\frac{1}{16a^2}t^2.$$

Proof. From Lemma 2.4 it is easy to see that

$$r(t) - r(t_c^+) \leqslant r(t) - r(t'), \tag{8}$$

where $t' = \sqrt{2}a$.
Note that

$$r'(t') = -\mathrm{Im}\, f'(u\varkappa + i\,t') = \mathrm{Im}\, s(u\varkappa + i\,t') - \frac{t'}{a^2} \leqslant \frac{1}{t'} - \frac{t'}{a^2} \leqslant 0. \tag{9}$$

For $t_0 \geqslant t'$ we have

$$\begin{aligned} r''(t_0) &= \mathrm{Re}\, s'(u\varkappa + i\,t_0) - \frac{1}{a^2} \\ &\leqslant \frac{|s(u\varkappa + i\,t_0)|}{\left|\sqrt{(u\varkappa + i\,t_0)^2 - 1}\right|} - \frac{1}{a^2} \leqslant \frac{1}{t_0^2} - \frac{1}{a^2} \leqslant -\frac{1}{2a^2}. \end{aligned} \tag{10}$$

Using (8)-(10) we get

$$r(t) - r(t_c^+) \leqslant (t - t')r'(t') - \frac{(t-t')^2}{2} \inf_{t_0 \geqslant t'} |r''(t_0)| \leqslant -\frac{1}{4a^2}(t-t')^2 \leqslant -\frac{1}{16a^2}t^2,$$

for $t \geqslant 2t'$.
The Lemma is proved. □

3 A bound for $n\,|\Delta_n(w) - \Delta_n(z)|$

We shall follow the notation of [11]. Let $\mathbf{W}(k)$ be the matrix obtained from $\mathbf{W}$ by deleting the kth row and kth column, and let

$$\mathbf{a}_k' = \left(W_{1k}, \ldots, W_{(k-1)k}, W_{(k+1)k}, \ldots, W_{nk}\right).$$

Denote by $F_n(x)$ the spectral distribution function of the matrix $\frac{1}{\sqrt{n}}\mathbf{W}$ and by $G(x)$ the distribution function of the semi-circle law with the density

$$g(x) = \frac{2}{\pi}\sqrt{(1-x^2)_+}.$$

Introduce the Stieltjes transforms of the distribution function $G(x)$ and of the expected spectral distribution function $\mathbf{E}\, F_n(x)$

$$s(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} dG(x) = -2z + 2\sqrt{z^2-1}, \quad s_n(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} d\mathbf{E}\, F_n(x),$$

respectively. Set

$$\mathbf{R}(z) := \left(\frac{1}{\sqrt{n}}\mathbf{W} - z\,\mathbf{I}_n\right)^{-1},$$

$$\varepsilon_k = \frac{1}{\sqrt{n}}W_{kk} - \frac{1}{n}\overline{\mathbf{a}}'_k\left(\frac{1}{\sqrt{n}}\mathbf{W}(k) - z\mathbf{I}_{n-1}\right)^{-1}\mathbf{a}_k + \frac{s_n(z)}{4},$$

where $\mathbf{I}_{n-1}$ denotes the $(n-1)\times(n-1)$ identity matrix, and $\overline{a}$ denotes the complex conjugate to a complex number a. Define the quantity

$$\delta_n(z) = -\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,\varepsilon_k\frac{1}{(z+\frac{s_n(z)}{4})(z+\frac{s_n(z)}{4}-\varepsilon_k)}.$$

Let

$$\Omega_0 := \left\{(y_1,\dots,y_n) : \frac{1}{n}\left|\mathrm{Tr}\,\mathbf{R}(z) - \mathbf{E}\,\mathrm{Tr}\,\mathbf{R}(z)\right| \leqslant \frac{1}{n^{\frac{7}{8}}\mathrm{Im}\,z\,|2z+s(z)|\,|4z+s_n(z)|}\right\}.$$

It was proved in [9] (see Theorem 1), for $\mathrm{Im}\,z \geqslant Cn^{-1/2}$, that

$$\frac{1}{n^8}\mathbf{E}\,\left|\mathrm{Tr}\,\mathbf{R}(z) - \mathbf{E}\,\mathrm{Tr}\,\mathbf{R}(z)\right|^8 \leqslant \frac{C}{n^8(\mathrm{Im}\,z)^8\,|2z+s(z)|^8\,|4z+s_n(z)|^8}.$$

Hence

$$\mathbf{P}\left(\overline{\Omega_0}\right) \leqslant \frac{C}{n}. \tag{11}$$

Lemma 3.1. Let $(y_1,\dots,y_n)\in\Omega_0$, and u_0, v_1, v_2 are real numbers, v_1, $v_2 \geqslant Cn^{-\frac{3}{8}}$. Then there exist positive constants C_1, $C_2>0$ such that we have

$$\left|\int_{v_1}^{v_2}|s(u_0+i\,s) - s_n(u_0+i\,s)|\,ds\right| \leqslant \frac{C_1}{n^{\frac{13}{16}}},$$

$$\left|\int_{v_1}^{v_2}\left|\frac{1}{n}\mathrm{Tr}\,\mathbf{R}(u_0+i\,s) - \frac{1}{n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{R}(u_0+i\,s)\right|ds\right| \leqslant \frac{C_2}{n^{\frac{11}{16}}}.$$

Proof. In [11] (see (4.14), (4.24), (5.64)-(5.65)) it was shown that there exists some positive constant $C_1>0$ such that, for $s \geqslant C\,n^{-1/2}$,

$$|s(u_0+i\,s) - s_n(u_0+i\,s)| \leqslant \frac{C_1}{ns^{3/2}}.$$

Since $(y_1,\dots,y_n)\in\Omega_0$, we obtain

$$\left|\frac{1}{n}\mathrm{Tr}\,\mathbf{R}(u_0+i\,s) - \frac{1}{n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{R}(u_0+i\,s)\right| \leqslant \frac{C_2}{n^{\frac{7}{8}}\,s^{\frac{3}{2}}}.$$

From the last two inequalities one can get

$$\left|\int_{v_1}^{v_2}|s(u_0+i\,s) - s_n(u_0+i\,s)|\,ds\right| \leqslant \int_{Cn^{-\frac{3}{8}}}^{+\infty}\frac{C_1}{ns^{3/2}}ds \leqslant \frac{C_1}{n^{\frac{13}{16}}},$$

$$\left|\int_{v_1}^{v_2}\left|\frac{1}{n}\mathrm{Tr}\,\mathbf{R}(u_0+i\,s) - \frac{1}{n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{R}(u_0+i\,s)\right|ds\right| \leqslant \int_{Cn^{-\frac{3}{8}}}^{+\infty}\frac{C_2}{n^{\frac{7}{8}}s^{3/2}}ds \leqslant \frac{C_2}{n^{\frac{11}{16}}}.$$

□

Lemma 3.2. Let $z = u_0 + i\,\frac{2a^2}{\sqrt{1+4a^2}}$. Then the following inequalities hold

$$|s(z) - s_n(z)| \leqslant \frac{\sqrt{1+4a^2}}{\sqrt{1+4a^2}-1}\ |\delta_n(z)|\,, \tag{12}$$

$$|4z + s_n(z)| \geqslant \frac{1}{2}\ |4z + s(z)|\,. \tag{13}$$

Proof. Put

$$\begin{aligned}
K_1(t) &= \frac{1+2a^2}{\sqrt{1+4a^2}} - t,\ -\infty < t \leqslant 0,\\
K_2(t) &= S(\sqrt{1+4a^2}\ e^{i\,t}),\ 0 \leqslant t \leqslant \pi,\\
K_3(t) &= -\frac{1+2a^2}{\sqrt{1+4a^2}} - t,\ 0 < t \leqslant +\infty,\\
K &= K_1(t) + K_2(t) + K_3(t).
\end{aligned}$$

Consider the function $4z + s(z)$. It is easy to see that

$$4z + s(z) = 2z + 2\sqrt{z^2-1} = 2S^{-1}(z),$$

and therefore

$$|4K_2(t) + s(K_2(t))| = \left|2\sqrt{1+4a^2}\ e^{i\,t}\right| = 2\sqrt{1+4a^2},\ 0 \leqslant t \leqslant \pi.$$

On the other hand,

$$|4K_{1,3}(t) + s(K_{1,3}(t))| \geqslant 2\frac{1+2a^2}{\sqrt{1+4a^2}} + 2\sqrt{\left(\frac{1+2a^2}{\sqrt{1+4a^2}}\right)^2 - 1} = 2\sqrt{1+4a^2}.$$

From the two last inequalities by the maximum principle we have

$$|4z + s(z)| \geqslant 2\sqrt{1+4a^2}, \tag{14}$$

for $\operatorname{Im} z \geqslant \frac{2a^2}{\sqrt{1+4a^2}}$.
Further, it can be checked that

$$s_n(z) - s(z) = \frac{4}{(4z+s(z))(4z+s_n(z))}(s_n(z) - s(z)) + \delta_n(z).$$

Noting that $|4z+s_n(z)| \geqslant 2$, for $\operatorname{Im} z \geqslant Cn^{-1/2}$ (see [11], (5.59)), and taking into account (14), one obtains

$$|s_n(z) - s(z)| \leqslant \frac{1}{\sqrt{1+4a^2}}\,|s_n(z) - s(z)| + |\delta_n(z)|\,,\ \text{for } \operatorname{Im} z \geqslant \frac{2a^2}{\sqrt{1+4a^2}}.$$

Solving this inequality with respect to $|s_n(z) - s(z)|$, we get

$$|s_n(z) - s(z)| \leqslant \frac{\sqrt{1+4a^2}}{\sqrt{1+4a^2}-1}\ |\delta_n(z)|\,,\ \text{for } \operatorname{Im} z \geqslant \frac{2a^2}{\sqrt{1+4a^2}}.$$

Thus, (12) follows. Now we will prove assertion (13).
From inequalities (12), (14) one can obtain

$$
\begin{aligned}
|4z+s_n(z)| &\geqslant |4z+s(z)| - |s(z)-s_n(z)| \geqslant |4z+s(z)| - \frac{\sqrt{1+4a^2}}{\sqrt{1+4a^2}-1}\,|\delta_n(z)| \\
&\geqslant |4z+s(z)|\left(1-\frac{|\delta_n(z)|}{\sqrt{1+4a^2}-1}\right). \qquad (15)
\end{aligned}
$$

It was proved in [11] (see Lemma 5.5) that $|\delta_n(z)| \leqslant \frac{C}{n\,\mathrm{Im}\,z}$. From this fact and (15) it follows, for $n \geqslant \frac{C\sqrt{1+4a^2}}{2a^2(\sqrt{1+4a^2}-1)}$, that

$$
|4z+s_n(z)| \geqslant \frac{1}{2}\,|4z+s(z)|\,.
$$

□

Lemma 3.3. There exists a positive constant $C(a)>0$ such that we have

$$
\int_{-\infty}^{+\infty}\left|s\left(t+i\,\frac{2a^2}{\sqrt{1+4a^2}}\right)-s_n\left(t+i\,\frac{2a^2}{\sqrt{1+4a^2}}\right)\right|dt \leqslant \frac{C(a)}{n}.
$$

Proof. It can be shown (see [11], Lemmas 4.3, 5.5) that, for $\mathrm{Im}\,z \geqslant C_0 n^{-1/2}$, the following inequality holds

$$
|\delta_n(z)| \leqslant \frac{C}{n\,\mathrm{Im}\,z}\;\frac{1}{|4z+s_n(z)|^2}.
$$

Let $z = t+i\,\frac{2a^2}{\sqrt{1+4a^2}}$. The last inequality implies

$$
\begin{aligned}
\int_{-\infty}^{+\infty}|\delta_n(z)|\,dt &\leqslant \frac{C}{n\,\mathrm{Im}\,z}\int_{-\infty}^{+\infty}\frac{1}{|4z+s_n(z)|^2}dt \\
&\leqslant \frac{C}{n\,\mathrm{Im}\,z}\left[\int_{-\infty}^{+\infty}|s_n(z)|^2\,dt+\int_{-\infty}^{+\infty}|\delta_n(z)|^2\,dt\right] \\
&\leqslant \frac{C}{n\,\mathrm{Im}\,z}\left[\mathbf{E}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{(x-t)^2+(\mathrm{Im}\,z)^2}dt\,dF_n(x)+\frac{C}{n\,\mathrm{Im}\,z}\int_{-\infty}^{+\infty}|\delta_n(z)|\,dt\right] \\
&\leqslant C_1\frac{1+4a^2}{n^2\,a^4}\int_{-\infty}^{+\infty}|\delta_n(z)|\,dt + C_2\frac{1+4a^2}{n\,a^4}.
\end{aligned}
$$

From here we get, for $n \geqslant \sqrt{2C_1}\,\frac{\sqrt{1+4a^2}}{a^2}$,

$$
\int_{-\infty}^{+\infty}|\delta_n(z)|\,dt \leqslant 2\,C_2\,\frac{1+4a^2}{n\,a^4}.
$$

Using this inequality and Lemma 3.2, we prove the Lemma. □

Lemma 3.4. Let $(y_1,\ldots,y_n)\in\Omega_0$. There exists a positive constant $C(a)>0$ such that we have

$$
\int_{-\infty}^{+\infty}\left|\frac{1}{n}\mathrm{Tr}\,\mathbf{R}\left(t+i\,\frac{2a^2}{\sqrt{1+4a^2}}\right)-\frac{1}{n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{R}\left(t+i\,\frac{2a^2}{\sqrt{1+4a^2}}\right)\right|dt \leqslant \frac{C(a)}{n^{\frac{7}{8}}}.
$$

Proof. Let $z = t + i\,\frac{2a^2}{\sqrt{1+4a^2}}$. By the definition of Ω_0 and Lemma 3.2, we conclude that

$$\int_{-\infty}^{+\infty} \left|\frac{1}{n}\operatorname{Tr}\mathbf{R}\left(t + i\,\frac{2a^2}{\sqrt{1+4a^2}}\right) - \frac{1}{n}\mathbf{E}\operatorname{Tr}\mathbf{R}\left(t + i\,\frac{2a^2}{\sqrt{1+4a^2}}\right)\right| dt$$
$$\leqslant \frac{2}{n^{\frac{7}{8}}\operatorname{Im} z}\int_{-\infty}^{+\infty} \frac{1}{|2z + s(z)|\,|4z + s(z)|} dt$$
$$\leqslant \frac{1}{2n^{\frac{7}{8}}\operatorname{Im} z}\int_{-\infty}^{+\infty} \frac{1}{\left|\sqrt{z^2-1}\right|\,\left|z + \sqrt{z^2-1}\right|} dt$$

$$\leqslant \frac{1}{n^{\frac{7}{8}}\operatorname{Im} z}\int_{0}^{+\infty} \frac{dt}{|z^2-1|} \leqslant \frac{1}{n^{\frac{7}{8}}\operatorname{Im} z}\left(\int_0^{\sqrt{2}} \frac{dt}{\sqrt{2}\operatorname{Im} z} + \int_{\sqrt{2}}^{+\infty} \frac{2}{t^2}\,dt\right) \leqslant \frac{C(a)}{n^{\frac{7}{8}}}.$$

□

Lemma 3.5. Let $(y_1, \ldots, y_n) \in \Omega_0$, $z \in \gamma^+$, $w \in \Gamma^+$. There exists a positive constant $C(a) > 0$ such that for all $\operatorname{Im} w \geqslant Cn^{-\frac{3}{8}}$ we have

$$n\,|\Delta_n(w) - \Delta_n(z)| \leqslant C(a)\,n^{\frac{5}{16}}.$$

Proof. From the definition of $\Delta_n(z)$ we have

$$|\Delta_n(w) - \Delta_n(z)| = \left|\int_z^w \left(s(\xi) - \frac{1}{n}\operatorname{Tr}\mathbf{R}(\xi)\right) d\xi\right|$$
$$\leqslant \left|\int_{\operatorname{Re} z}^{\operatorname{Re} w} \left|s(t + i\,\frac{2a^2}{\sqrt{1+4a^2}}) - s_n(t + i\,\frac{2a^2}{\sqrt{1+4a^2}})\right| dt\right|$$
$$+ \left|\int_{\operatorname{Im} z}^{\frac{2a^2}{\sqrt{1+4a^2}}} |s(\operatorname{Re} z + i\,s) - s_n(\operatorname{Re} z + i\,s)|\,ds\right|$$
$$+ \left|\int_{\frac{2a^2}{\sqrt{1+4a^2}}}^{\operatorname{Im} w} |s(\operatorname{Re} w + i\,s) - s_n(\operatorname{Re} w + i\,s)|\,ds\right|$$

$$+ \left|\int_{\operatorname{Re} z}^{\operatorname{Re} w} \left|\frac{1}{n}\operatorname{Tr}\mathbf{R}(t + i\,\frac{2a^2}{\sqrt{1+4a^2}}) - \frac{1}{n}\mathbf{E}\operatorname{Tr}\mathbf{R}(t + i\,\frac{2a^2}{\sqrt{1+4a^2}})\right| dt\right|$$
$$+ \left|\int_{\operatorname{Im} z}^{\frac{2a^2}{\sqrt{1+4a^2}}} \left|\frac{1}{n}\operatorname{Tr}\mathbf{R}(\operatorname{Re} z + i\,s) - \frac{1}{n}\mathbf{E}\operatorname{Tr}\mathbf{R}(\operatorname{Re} z + i\,s)\right| ds\right|$$
$$+ \left|\int_{\frac{2a^2}{\sqrt{1+4a^2}}}^{\operatorname{Im} w} \left|\frac{1}{n}\operatorname{Tr}\mathbf{R}(\operatorname{Re} w + i\,s) - \frac{1}{n}\mathbf{E}\operatorname{Tr}\mathbf{R}(\operatorname{Re} w + i\,s)\right| ds\right|.$$

The Lemma follows from Lemmas 3.1, 3.3, and 3.4. □

4 Bounds for the tails

Let

$$\varepsilon = \min\left\{\frac{\sqrt{1+4a^2}}{4},\ \frac{a^6}{16(1+4a^2)^2}\right\}\frac{p(u)}{1+4a^2}, \quad \gamma_*^\pm = \sum_{j\neq 3}\gamma_j^\pm, \quad \Gamma_1^\pm = \Gamma_{11}^\pm + \Gamma_{12}^\pm,$$

where

$$\Gamma_{11}^+(t) = \operatorname{Re} z_c^+ + i\,t, \quad 0 \leqslant t \leqslant Cn^{-\frac{3}{8}},$$

$$\Gamma_{12}^+(t) = \operatorname{Re} z_c^+ + i\,t, \quad Cn^{-\frac{3}{8}} \leqslant t \leqslant \operatorname{Im} z_c^+ - \varepsilon,$$

$$\Gamma_{11}^-(t) = \overline{\Gamma_{11}^+(t)}, \quad \Gamma_{12}^-(t) = \overline{\Gamma_{12}^+(t)}.$$

Consider the quantities

$$\begin{aligned}
I_1^{bd} &= n\int_{\gamma_*^b}\frac{dz}{2\pi}\int_{\Gamma_3^d}\frac{dw}{2\pi}g_n(z,w)\exp\left\{n\big(f_n(w)-f_n(z)\big)\right\},\\
I_2^{bd} &= n\int_{\gamma^b}\frac{dz}{2\pi}\int_{\Gamma_{12}^d+\Gamma_2^d}\frac{dw}{2\pi}g_n(z,w)\exp\left\{n\big(f_n(w)-f_n(z)\big)\right\},\\
I_3^{bd} &= n\int_{\gamma^b}\frac{dz}{2\pi}\int_{\Gamma_{11}^d}\frac{dw}{2\pi}g_n(z,w)\exp\left\{n\big(f_n(w)-f_n(z)\big)\right\},
\end{aligned}$$

where $b, d \in \{+,-\}$.

Theorem 4.1. Let $(y_1,\dots,y_n) \in \Omega_0$, $1+4a^2-u^2 \geqslant cn^{-\frac{1}{3}}$. For all $b, d \in \{+,-\}$, $i = 1,2,3$, we have

$$\left|I_i^{bd}\right| \leqslant C_1(a)\exp\{-n\,C_2(a)\,p^4(u)\}.$$

To prove this theorem we need the next

Lemma 4.2. Let $1+4a^2-u^2 \geqslant cn^{-\frac{1}{3}}$, $z \in \gamma^+$, $w \in \Gamma_{11}^+$. Then we have, for some $C(a) > 0$,

$$|z-w| \geqslant C(a)\,n^{-\frac{1}{6}}.$$

Proof. Using the definitions of contours γ^+, Γ_{11}^+ and quantity δ, we get

$$\begin{aligned}
|z-w| &\geqslant \operatorname{Im} z - \operatorname{Im} w \geqslant \frac{2a^2}{\sqrt{1+4a^2}}\sin\delta - Cn^{-\frac{3}{8}}\\
&\geqslant C_1(a)\,n^{-\frac{1}{6}} - C_2(a)\,n^{-\frac{3}{8}} \geqslant C(a)\,n^{-\frac{1}{6}}.
\end{aligned}$$

□

Proof of Theorem 4.1. We shall give the proof only for the integral $\left|I_3^{++}\right|$. The other cases are similar.

Let $w = u\varkappa + i\,t$, $w' = u\varkappa + i\,t'$, $\varkappa = \frac{1+2a^2}{1+4a^2}$, $t' = Cn^{-\frac{3}{8}}$, $t \in [0, t']$. Using integration by parts, one obtains

$$I_3^{++} = \frac{n}{4a^2\pi^2}\int_{\gamma^+} dz \int_{\Gamma_{11}^+} dw \left(\frac{z}{z-w}f_n'(z) + \frac{1}{n}\frac{z}{(z-w)^2}\right)\exp\left\{n\big(f_n(w) - f_n(z)\big)\right\}$$
$$- \frac{1}{4a^2\pi^2}\int_{\gamma^+}\left(\frac{w'}{z-w'}\,e^{n\,f_n(w')} - \frac{u\varkappa}{z-u\varkappa}\,e^{n\,f_n(u\varkappa)}\right)\,e^{-n\,f_n(z)}dz.$$

The last equality implies

$$\left|I_3^{++}\right| \leqslant \frac{n}{4a^2\pi^2}\int_{\gamma^+}|dz|\int_{\Gamma_{11}^+}|dw|\left|\frac{z}{z-w}f_n'(z) + \frac{1}{n}\frac{z}{(z-w)^2}\right|\exp\left\{n\mathrm{Re}\big(f_n(w) - f_n(z)\big)\right\}$$
$$+ \frac{1}{4a^2\pi^2}\int_{\gamma^+}\left(\left|\frac{w'}{z-w'}\right| + \left|\frac{u\varkappa}{z-u\varkappa}\right|\,e^{n\mathrm{Re}\left(f_n(u\varkappa)-f_n(w')\right)}\right)\exp\left\{n\mathrm{Re}\big(f_n(w') - f_n(z)\big)\right\}|dz|.$$

Note that for all $w \in \Gamma_{11}^+$ we have inequality

$$\mathrm{Re}\big(f_n(w) - f_n(w')\big) \leqslant \frac{t'^2}{2a^2}.$$

From this fact and Lemma 4.2 we get, for large enough $n \geqslant n(a)$,

$$\left|I_3^{++}\right| \leqslant C(a)\,n\int_{\gamma^+}\left(|z\,f_n'(z)| + |z| + 1\right)\,e^{\frac{n\,t'^2}{2a^2}}\exp\left\{n\mathrm{Re}\big(f_n(w') - f_n(z)\big)\right\}|dz|.$$

Since $\mathrm{Im}\,z \geqslant C(a)\,n^{-\frac{1}{6}}$, it follows that

$$|f_n'(z)| \leqslant C(a)\,n^{\frac{1}{6}}(|z| + 1),$$

for some constant $C(a) > 0$ and large enough n. The two last inequalities yield

$$\left|I_3^{++}\right| \leqslant C(a)\,n^{\frac{7}{6}}\exp\left\{\frac{n\,t'^2}{2a^2}\right\}\int_{\gamma^+}(|z|+1)^2\exp\left\{n\mathrm{Re}\big(f_n(w') - f_n(z)\big)\right\}|dz|$$
$$\leqslant C_1(a)\,n^{\frac{7}{6}}\exp\left\{\frac{n\,t'^2}{2a^2}\right\}\int_{\gamma_2^+ + \gamma_3^+ + \gamma_4^+}\exp\left\{n\mathrm{Re}\big(f(w') - f(z_c^+)\big)\right\}$$
$$\times\exp\left\{-n\mathrm{Re}\big(f(z) - f(z_c^+)\big)\right\}\exp\left\{n\mathrm{Re}\big(\Delta_n(w') - \Delta_n(z)\big)\right\}|dz|$$
$$+ C_2(a)\,n^{\frac{7}{6}}\exp\left\{\frac{n\,t'^2}{2a^2}\right\}\int_{\gamma_1^+ + \gamma_5^+}(|\mathrm{Re}\,z| + C_3(a))^2\exp\left\{n\mathrm{Re}\big(f(w') - f(z_c^+)\big)\right\}$$
$$\times\exp\left\{-n\mathrm{Re}\big(f(z) - f(z_c^+)\big)\right\}\exp\left\{n\mathrm{Re}\big(\Delta_n(w') - \Delta_n(z)\big)\right\}|dz|\,.$$

Using Lemmas 2.1, 2.6, 3.5, we get

$$\left|I_3^{++}\right| \leqslant C_1(a)\exp\left\{-C_2(a)\,n\,p^4(u)\right\}$$
$$\times\left(1 + \int_{\gamma_1^+ + \gamma_5^+}(|\mathrm{Re}\,z| + C_3(a))^2\exp\left\{-n\mathrm{Re}\big(f(z) - f(z_c^+)\big)\right\}|dz|\right)$$
$$\leqslant C_1(a)\exp\left\{-C_2(a)\,n\,p^4(u)\right\}\left(1 + C_3(a)\int_{-\infty}^{0}(C_4(a) - t)^2\right.$$
$$\left.\times\left(e^{-n\mathrm{Re}\left(f(S(\omega_\delta - t)) - f(z_c^+)\right)} + e^{-n\mathrm{Re}\left(f(S(\omega_{\pi-\delta} + t)) - f(z_c^+)\right)}\right)dt\right),$$

where $\omega_\delta = \sqrt{1+4a^2}\, e^{i\,\delta}$, $\omega_{\pi-\delta} = \sqrt{1+4a^2}\, e^{i\,(\pi-\delta)}$.
The last inequality and Lemmas 2.2, 2.3 imply

$$\left|I_3^{++}\right| \leqslant C_1(a)\exp\Big\{-C_2(a)\,n\,p^4(u)\Big\}.$$

Theorem 4.1 is proved. □

5 Critical points for $f_n(z)$

Lemma 5.1. Let $(y_1,\ldots,y_n)\in\Omega_0$. There are critical points $z_*^\pm$ for $f_n(z)$ such that

$$\left|z_*^\pm - z_c^\pm\right| \leqslant \frac{C(a)}{n^{\frac{7}{8}}p^2(u)}.$$

Proof. We shall prove for the "plus" case only.
It is easily seen that

$$f_n'(z) = f'(z) + \left(s(z) - \frac{1}{n}\mathrm{Tr}\,\mathbf{R}(z)\right).$$

Thus, it is sufficient to show the inequality

$$|f'(z)| > \left|s(z) - \frac{1}{n}\mathrm{Tr}\,\mathbf{R}(z)\right|,$$

for

$$\left|z - z_c^+\right| = \frac{2(1+4a^2)^2}{a^2}\frac{1}{n^{\frac{7}{8}}p^2(u)}.$$

Then the Lemma follows from Rouche's theorem (see [15]).
Let

$$\overline{D} = \left\{\xi : \left|\xi - z_c^+\right| \leqslant \frac{2(1+4a^2)^2}{a^2}\frac{1}{n^{\frac{7}{8}}p^2(u)}\right\},$$

$$z\in\partial\overline{D},$$

where $\partial\overline{D}$ denotes the boundary of domain $\overline{D}$. We have

$$\begin{aligned}|f'(z)| &\geqslant \left|\frac{z-z_c^+}{a^2} - s'(z_c^+)(z-z_c^+)\right| - \sup_{\xi\in\overline{D}}|s''(\xi)|\,\frac{\left|z-z_c^+\right|^2}{2}\\ &= \frac{\sqrt{1+4a^2}\,p(u)}{a^2\sqrt{4a^4+p^2(u)}}\left|z-z_c^+\right| - \sup_{\xi\in\overline{D}}\frac{1}{\left|\xi^2-1\right|^{\frac{3}{2}}}\left|z-z_c^+\right|^2.\end{aligned}$$

Noting that

$$\left|\xi^2-1\right| \geqslant \left|{z_c^+}^2-1\right| - \left|{z_c^+}^2-\xi^2\right| \geqslant \frac{4a^4}{1+4a^2} - C(a)\left|z-z_c^+\right| \geqslant \frac{2a^4}{1+4a^2}, \tag{16}$$

for all $\xi\in\overline{D}$ and large enough n, we get

$$|f'(z)| \geqslant \frac{\sqrt{1+4a^2}\,p(u)}{2a^2(1+2a^2)}\left|z-z_c^+\right| \geqslant \frac{(1+4a^2)^{\frac{3}{2}}}{a^4\,n^{\frac{7}{8}}\,p(u)}. \tag{17}$$

It can be shown (see [11], (4.14), (4.24), (5.61)) that

$$|s(z)-s_n(z)| \leqslant \frac{C(a)}{n\,\mathrm{Im}\,z} \leqslant \frac{C(a)}{n\,p(u)},$$

for large n and $z \in \overline{D}$. Combining the last inequality, the definition of Ω_0, and (16), we conclude

$$\left|s(z)-\frac{1}{n}\mathrm{Tr}\,\mathbf{R}(z)\right| \leqslant |s(z)-s_n(z)| + \left|\frac{1}{n}\mathrm{Tr}\,\mathbf{R}(z)-\frac{1}{n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{R}(z)\right|$$
$$\leqslant \frac{C(a)}{n\,p(u)} + \frac{1}{2\,|z^2-1|^{\frac{1}{2}}\,n^{\frac{7}{8}}\,\mathrm{Im}\,z} \leqslant \frac{(1+4a^2)^{\frac{3}{2}}}{2\sqrt{2}\,a^4\,n^{\frac{7}{8}}\,p(u)}\left(1+\frac{C(a)}{n^{\frac{1}{8}}}\right) \leqslant \frac{(1+4a^2)^{\frac{3}{2}}}{2\,a^4\,n^{\frac{7}{8}}\,p(u)}.$$

This inequality and (17) yield the Lemma. □

6 The main term

Let $\varkappa = \frac{1+2a^2}{1+4a^2}$, $v=\sqrt{1+4a^2}$, $\hat{g}_n(\theta,t) = g_n\big(S(ve^{i\theta}),\, u\varkappa + i\,t\big)\Big(ve^{i\theta}-\frac{1}{v}e^{-i\theta}\Big)$, $\hat{f}_n(\theta) = f_n(S(ve^{i\theta}))$, $\hat{f}(\theta)=f(S(ve^{i\theta}))$.
It is easy to see that

$$f_n(u\varkappa+i\,t)-f_n(S(ve^{i\theta})) = -h^2(u)\frac{(t-t_c^+)^2}{2}$$
$$+\,h^2(u)\left(ve^{i\,\theta_c}-\frac{1}{v}e^{-i\,\theta_c}\right)^2\frac{(\theta-\theta_c)^2}{8}+\Theta_n^{\Gamma}(u)+\Theta_n^{\gamma}(u),$$

where

$$h^2(u) = \frac{1}{a^2\,((1+2a^2)^2-u^2)}\big((1+2a^2)\,p^2(u)+i\,2a^2\,u\,p(u)\big),$$

$$\Theta_n^{\Gamma}(u) = i\,f_n'(z_c^+)(t-t_c^+)+\big(f''(z_c^+)-f_n''(z_c^+)\big)\,\frac{(t-t_c^+)^2}{2}-i\,f_n'''(u\varkappa+i\,t_\eta)\frac{(t-t_c^+)^3}{6},$$

$$\Theta_n^{\gamma}(u) = -\hat{f}_n'(\theta_c)(\theta-\theta_c)+\Big(\hat{f}''(\theta_c)-\hat{f}_n''(\theta_c)\Big)\,\frac{(\theta-\theta_c)^2}{2}-\hat{f}_n'''(\theta_\eta)\frac{(\theta-\theta_c)^3}{6},$$

$$t_\eta\in[t_c^+,t],\quad \theta_\eta\in[\theta_c,\theta].$$

Thus, we can write

$$\frac{n}{4\pi^2}\int_{\gamma_3^+}dz\int_{\Gamma_3^+} g_n(z,w)\,e^{n\{f_n(w)-f_n(z)\}}dw =$$
$$-\,\frac{n}{8\pi^2}\int_{\theta_c-\varepsilon}^{\theta_c+\varepsilon}d\theta\int_{t_c^+-\varepsilon}^{t_c^++\varepsilon}dt\,\hat{g}_n(\theta,t)\;e^{n\left(\Theta_n^{\Gamma}(u)+\Theta_n^{\gamma}(u)\right)}$$
$$\times\exp\left\{-n\,h^2(u)\frac{(t-t_c^+)^2}{2}\right\}\;\exp\left\{n\,h^2(u)\left(ve^{i\,\theta_c}-\frac{1}{v}e^{-i\,\theta_c}\right)^2\frac{(\theta-\theta_c)^2}{8}\right\}dt\,d\theta$$

$$= -\frac{n}{8\pi^2}\sum_{s=0}^{k}\int_{\theta_c-\varepsilon}^{\theta_c+\varepsilon}\int_{t_c^+-\varepsilon}^{t_c^++\varepsilon}\hat{g}_n(\theta,t)\frac{n^s}{s!}\left(\Theta_n^\Gamma(u)+\Theta_n^\gamma(u)\right)^s$$

$$\times\exp\left\{-n\,h^2(u)\frac{(t-t_c^+)^2}{2}\right\}\,\exp\left\{n\,h^2(u)\left(ve^{i\theta_c}-\frac{1}{v}e^{-i\theta_c}\right)^2\frac{(\theta-\theta_c)^2}{8}\right\}dt\,d\theta$$

$$-\frac{n}{8\pi^2}\int_{\theta_c-\varepsilon}^{\theta_c+\varepsilon}\int_{t_c^+-\varepsilon}^{t_c^++\varepsilon}\hat{g}_n(\theta,t)\left[e^{n\left(\Theta_n^\Gamma(u)+\Theta_n^\gamma(u)\right)}-\sum_{s=0}^{k}\frac{n^s\left(\Theta_n^\Gamma(u)+\Theta_n^\gamma(u)\right)^s}{s!}\right]$$

$$\times\exp\left\{-n\,h^2(u)\frac{(t-t_c^+)^2}{2}\right\}\,\exp\left\{n\,h^2(u)\left(ve^{i\theta_c}-\frac{1}{v}e^{-i\theta_c}\right)^2\frac{(\theta-\theta_c)^2}{8}\right\}dt\,d\theta.$$

Note that

$$\operatorname{Re}\left\{h^2(u)\left(ve^{i\theta_c}-\frac{1}{v}e^{-i\theta_c}\right)^2\right\}=-\frac{4}{a^2}\varkappa\,p^2(u).$$

Lemma 6.1. Let $(y_1,\dots,y_n)\in\Omega_0$, $|\operatorname{Im} z|\geqslant C(a)\,p(u)$. Then we have, for any $s\geqslant 1$,

$$|f_n'(z)-f'(z)|\leqslant\frac{C(a)}{n^{\frac{7}{8}}p(u)},\qquad \mathbf{E}\,|f_n'(z)-f'(z)|^s\leqslant\frac{C(a)}{n^s\,p^s(u)},$$

$$|f_n''(z)-f''(z)|\leqslant\frac{C(a)}{n^{\frac{7}{8}}p^2(u)},\qquad \mathbf{E}\,|f_n''(z)-f''(z)|^s\leqslant\frac{C(a)}{n^s\,p^{2s}(u)},$$

$$|f_n'''(z)-f'''(z)|\leqslant\frac{C(a)}{n^{\frac{7}{8}}p^3(u)},\qquad \mathbf{E}\,|f_n'''(z)-f'''(z)|^s\leqslant\frac{C(a)}{n^s\,p^{3s}(u)}.$$

Proof. The first inequality was proved in Lemma 5.1. The second one follows from the proof of Lemma 5.1 and inequality $\frac{1}{n^s}\mathbf{E}\,|\operatorname{Tr}\mathbf{R}(z)-\mathbf{E}\operatorname{Tr}\mathbf{R}(z)|^s\leqslant\frac{C}{n^s\,(\operatorname{Im} z)^s}$ (see [9], Theorem 1). Cauchy's theorem and the first two inequalities imply the other inequalities. □

Lemma 6.2. Let $(y_1,\dots,y_n)\in\Omega_0$, $1+4a^2-u^2\geqslant c\,n^{-\frac{1}{3}+\nu}$. For any integer $r\geqslant 1$ there exists some $k_0(\nu,r)$ such that, for all $k\geqslant k_0(\nu,r)$, we have

$$\frac{n}{8\pi^2}\int_{\theta_c-\varepsilon}^{\theta_c+\varepsilon}\int_{t_c^+-\varepsilon}^{t_c^++\varepsilon}|\hat{g}_n(\theta,t)|\left|e^{n\left(\Theta_n^\Gamma(u)+\Theta_n^\gamma(u)\right)}-\sum_{s=0}^{k}\frac{n^s\left(\Theta_n^\Gamma(u)+\Theta_n^\gamma(u)\right)^s}{s!}\right|$$

$$\times\exp\left\{-n\,\frac{1+2a^2}{a^2((1+2a^2)^2-u^2)}\frac{(t-t_c^+)^2}{2}p^2(u)\right\}\,\exp\left\{-n\,\frac{\varkappa}{a^2}\frac{(\theta-\theta_c)^2}{2}p^2(u)\right\}dt\,d\theta$$

$$\leqslant C(a,r)n^{-r}.$$

Proof. Note that for $|\theta-\theta_c|\leqslant\varepsilon$ and $|t-t_c^+|\leqslant\varepsilon$ we have

$$\left|\operatorname{Re}(\hat{f}'''(\theta))\right|\leqslant 5\,\frac{1+2a^2}{a^2}\,|\sin\theta|\leqslant 6\,\frac{1+2a^2}{a^2}p(u),\tag{18}$$

$$|\operatorname{Re}(i\,f'''(u\varkappa+i\,t))|=|\operatorname{Im} s''(u\varkappa+i\,t)|\leqslant 2\sqrt{2}\,\frac{(1+4a^2)^2}{a^8}\,p(u).\tag{19}$$

The last inequality follows from (7).
Using Lemmas 5.1, 6.1, one can show that

$$\begin{aligned}\left|f_n'(z_c^+)(t-t_c^+)\right| &= \left|f_n'(z_c^+)-f_n'(z_*^+)\right|\left|t-t_c^+\right| \\ &\leqslant \left|f''(z_c^+)\right|\left|z_*^+-z_c^+\right|\left|t-t_c^+\right| + \frac{C(a)}{n^{\frac{9}{8}}}, \\ \left|\hat{f}_n'(\theta_c)(\theta-\theta_c)\right| &\leqslant C_1(a)\left|f_n'(z_c^+)-f_n'(z_*^+)\right|\left|\theta-\theta_c\right| \\ &\leqslant C_1(a)\left|f''(z_c^+)\right|\left|z_*^+-z_c^+\right|\left|\theta-\theta_c\right| + \frac{C_2(a)}{n^{\frac{9}{8}}}.\end{aligned}$$

This implies

$$\left|f_n'(z_c^+)(t-t_c^+)\right| \leqslant \frac{C_1(a)}{n^{\frac{1}{24}}}\,p^2(u)\left|t-t_c^+\right|^2 + \frac{C_2(a)}{n^{\frac{9}{8}}}, \tag{20}$$

$$\left|\hat{f}_n'(\theta_c)(\theta-\theta_c)\right| \leqslant \frac{C_1(a)}{n^{\frac{1}{24}}}\,p^2(u)\left|\theta-\theta_c\right|^2 + \frac{C_2(a)}{n^{\frac{9}{8}}}, \tag{21}$$

for $|t-t_c^+| \geqslant \frac{n^{\frac{1}{24}}}{p(u)}\,|z_*^+-z_c^+|$, $|\theta-\theta_c| \geqslant \frac{n^{\frac{1}{24}}}{p(u)}\,|z_*^+-z_c^+|$, and

$$\left|f_n'(z_c^+)(t-t_c^+)\right| \leqslant \frac{C(a)}{n^{\frac{25}{24}}}, \qquad \left|\hat{f}_n'(\theta_c)(\theta-\theta_c)\right| \leqslant \frac{C(a)}{n^{\frac{25}{24}}}, \tag{22}$$

for $|t-t_c^+| \leqslant \frac{n^{\frac{1}{24}}}{p(u)}\,|z_*^+-z_c^+|$, $|\theta-\theta_c| \leqslant \frac{n^{\frac{1}{24}}}{p(u)}\,|z_*^+-z_c^+|$.
Using (18)-(22) and Lemma 6.1, we get, for $|\theta-\theta_c| \leqslant \varepsilon$, $|t-t_c^+| \leqslant \varepsilon$, and large enough n,

$$\begin{aligned}\left|\operatorname{Re}\Theta_n^\Gamma(u)\right| &- \frac{1+2a^2}{a^2((1+2a^2)^2-u^2)}\frac{(t-t_c^+)^2}{2}p^2(u) \\ &\leqslant -\frac{1}{a^2(1+2a^2)}\frac{(t-t_c^+)^2}{8}p^2(u) + \frac{C(a)}{n^{\frac{25}{24}}},\end{aligned} \tag{23}$$

$$\left|\operatorname{Re}\Theta_n^\gamma(u)\right| - \frac{\varkappa}{a^2}\frac{(\theta-\theta_c)^2}{2}p^2(u) \leqslant -\frac{\varkappa}{a^2}\frac{(\theta-\theta_c)^2}{8}p^2(u) + \frac{C(a)}{n^{\frac{25}{24}}}. \tag{24}$$

Furthermore, it can be checked, for $|\theta-\theta_c| \leqslant \varepsilon$, and $|t-t_c^+| \leqslant \varepsilon$, that

$$|f'''(u\varkappa + i\,t)| \leqslant C_1(a), \qquad \left|\hat{f}_\theta'''(\theta)\right| \leqslant C_2(a),$$

and thus, we have

$$\left|\Theta_n^\Gamma(u)\right| \leqslant \frac{C_1(a)}{n^{\frac{25}{24}}} + \frac{C_2(a)}{n^{\frac{1}{24}}}p^2(u)\left|t-t_c^+\right|^2 + C_3(a)\left|t-t_c^+\right|^3, \tag{25}$$

$$\left|\Theta_n^\gamma(u)\right| \leqslant \frac{C_1(a)}{n^{\frac{25}{24}}} + \frac{C_2(a)}{n^{\frac{1}{24}}}p^2(u)\left|\theta-\theta_c\right|^2 + C_3(a)\left|\theta-\theta_c\right|^3. \tag{26}$$

Combining (23)-(26) and the fact that

$$\left|e^z - \sum_{s=0}^{k}\frac{z^s}{s!}\right| \leqslant \frac{2^{\frac{k+1}{2}}|z|^{k+1}}{(k+1)!}e^{|\operatorname{Re} z|},$$

for any complex z and integer $k \geqslant 1$, we get

$$\frac{n}{8\pi^2}\int_{\theta_c-\varepsilon}^{\theta_c+\varepsilon}\int_{t_c^+-\varepsilon}^{t_c^++\varepsilon}|\hat{g}_n(\theta,t)|\left|e^{n\left(\Theta_n^\Gamma(u)+\Theta_n^\gamma(u)\right)}-\sum_{s=0}^{k}\frac{n^s\,\left(\Theta_n^\Gamma(u)+\Theta_n^\gamma(u)\right)^s}{s!}\right|$$
$$\times\exp\left\{-n\,\frac{1+2a^2}{a^2((1+2a^2)^2-u^2)}\frac{(t-t_c^+)^2}{2}p^2(u)\right\}\,\exp\left\{-n\,\frac{\varkappa}{a^2}\frac{(\theta-\theta_c)^2}{2}p^2(u)\right\}dt\,d\theta$$
$$\leqslant C(a,k)n^{\frac{4}{3}}\int_{\theta_c-\varepsilon}^{\theta_c+\varepsilon}\int_{t_c^+-\varepsilon}^{t_c^++\varepsilon}\left|n\Theta_n^\Gamma(u)+n\Theta_n^\gamma(u)\right|^{k+1}e^{-C_1(a)\,n\,p^2(u)(t-t_c^+)^2-C_2(a)\,n\,p^2(u)(\theta-\theta_c)^2}$$
$$\leqslant\frac{n^{\frac{1}{3}}}{p^2(u)}\int_0^{n\varepsilon^2p^2(u)}\int_0^{n\varepsilon^2p^2(u)}\left(\frac{C_1(a,k)}{n^{\frac{k+1}{24}}}+\frac{C_2(a,k)}{n^{\frac{k+1}{24}}}\rho^{k+1}+\frac{C_3(a,k)}{n^{\frac{k+1}{24}}}\sigma^{k+1}\right.$$
$$\left.+\frac{C_4(a,k)}{p^{3k+3}(u)n^{\frac{k+1}{2}}}\,\rho^{\frac{3k+3}{2}}+\frac{C_5(a,k)}{p^{3k+3}(u)n^{\frac{k+1}{2}}}\,\sigma^{\frac{3k+3}{2}}\right)e^{-\rho}e^{-\sigma}\,\frac{d\rho\,d\sigma}{\sqrt{\rho}\sqrt{\sigma}}$$
$$\leqslant C_1(a,k)\frac{n^{\frac{1}{3}}}{p^2(u)\,n^{\frac{k+1}{24}}}+C_2(a,k)\frac{n^{\frac{1}{3}}}{p^{3k+5}(u)\,n^{\frac{k+1}{2}}},$$

where $\rho=C_1(a)\,n\,p^2(u)(t-t_c^+)^2$, $\sigma=C_2(a)\,n\,p^2(u)(\theta-\theta_c)^2$.
From the last inequality the lemma follows. □

Now we get some representation of function $\hat{g}_n(\theta,t)$.

Lemma 6.3. Let $|\theta-\theta_c|\leqslant\varepsilon$, $|t-t_c|\leqslant\varepsilon$. Then there exist some positive constants $C_2(a)$, $C_3(a)$ and a function $C_1(\theta)\geqslant 0$ depending on θ such that

$$\hat{g}_n(\theta,t)=\frac{z_c^+}{a^2}f''(z_c^+)\left(ve^{i\,\theta_c}-\frac{1}{v}e^{-i\,\theta_c}\right)+\psi_1(\theta)\,C_1(\theta)(t-t_c^+)+\psi_2(a)\,C_2(a)(\theta-\theta_c)$$
$$+\psi_3(a,t,\theta)\,C_3(a)\Bigg(|f_n'(u\varkappa+i\,t)-f'(u\varkappa+i\,t)|+|f_n''(u\varkappa+i\,t)-f''(u\varkappa+i\,t)|$$
$$+|f_n'''(u\varkappa+i\,t)-f'''(u\varkappa+i\,t)|\,p(u)+|f_n''''(u\varkappa+i\,t)-f''''(u\varkappa+i\,t)|\,p^2(u)$$
$$+\left|t-t_c^+\right|^2+|\theta-\theta_c|^2+\left|t-t_c^+\right|\,|\theta-\theta_c|\Bigg),$$

where $|\psi_1(\theta)|$, $|\psi_2(a)|$, $|\psi_3(a,t,\theta)|\leqslant 1$.

Proof. Let $w=u\varkappa+i\,t$, $z=S(ve^{i\,\theta})$. A computation gives

$$g_n(\theta,t)=\frac{1}{a^2}f_n'(w)+\frac{z}{a^2}\left(f_n''(w)+f_n'''(w)\frac{z-w}{2}+f_n''''(\xi)\frac{(z-w)^2}{6}\right)=\frac{z_c^+}{a^2}f''(z_c^+)$$
$$+\frac{1}{a^2}\big(f_n'(w)-f'(w)\big)+\frac{1}{a^2}\big(f'(w)-f'(z_c^+)\big)+\frac{z}{a^2}\big(f_n''(w)-f''(w)\big)$$
$$+\frac{z}{a^2}\big(f''(w)-f''(z_c^+)\big)+\frac{1}{a^2}f''(z_c^+)(z-z_c^+)+\frac{z}{2a^2}\big(f_n'''(w)-f'''(w)\big)(z-w)$$
$$+\frac{z}{2a^2}\big(f'''(w)-f'''(z_c^+)\big)(z-w)+\frac{z}{2a^2}f'''(z_c^+)(z-w)$$
$$+\frac{z}{6a^2}\big(f_n''''(\xi)-f''''(\xi)\big)(z-w)^2+\frac{z}{6a^2}f''''(\xi)(z-w)^2,$$

where $|\xi - w| \leqslant |z-w|$. From here, using the expansions

$$f'(w) - f'(z_c^+) = i\, f''(z_c^+)(t-t_c^+) - f'''(\eta_1)\frac{(t-t_c^+)^2}{2}, \qquad \left|\eta_1 - z_c^+\right| \leqslant \left|w - z_c^+\right|,$$

$$f''(w) - f''(z_c^+) = i\, f'''(z_c^+)(t-t_c^+) - f''''(\eta_2)\frac{(t-t_c^+)^2}{2}, \qquad \left|\eta_2 - z_c^+\right| \leqslant \left|w - z_c^+\right|,$$

$$f'''(w) - f'''(z_c^+) = i\, f''''(\eta_3)(t-t_c^+), \qquad \left|\eta_3 - z_c^+\right| \leqslant \left|w - z_c^+\right|,$$

$$z - z_c^+ = \frac{i}{2}\left(ve^{i\,\theta_c} - \frac{1}{v}e^{-i\,\theta_c}\right)(\theta-\theta_c) - S(ve^{i\,\theta_\eta})\frac{(\theta-\theta_c)^2}{2}, \qquad |\theta_\eta - \theta_c| \leqslant |\theta-\theta_c|,$$

we get

$$\begin{aligned} g_n(\theta,t) = {} & \frac{z_c^+}{a^2}f''(z_c^+) + \frac{i}{a^2}\left(f''(z_c^+) + \frac{z}{2}f'''(z_c^+)\right)(t-t_c^+) \\ & + \frac{i}{2a^2}\left(f''(z_c^+) + \frac{z_c^+}{2}f'''(z_c^+)\right)\left(ve^{i\,\theta_c} - \frac{1}{v}e^{-i\,\theta_c}\right)(\theta-\theta_c) \\ & + \psi\, C(a)\Bigg(\, |f_n'(u\varkappa + i\,t) - f'(u\varkappa + i\,t)| + |f_n''(u\varkappa + i\,t) - f''(u\varkappa + i\,t)| \\ & + |f_n'''(u\varkappa + i\,t) - f'''(u\varkappa + i\,t)|\, p(u) + |f_n''''(u\varkappa + i\,t) - f''''(u\varkappa + i\,t)|\, p^2(u) \\ & + \left|t-t_c^+\right|^2 + |\theta-\theta_c|^2 + \left|t-t_c^+\right|\,|\theta-\theta_c|\Bigg), \end{aligned}$$

for some $|\psi| \leqslant 1$. This expression and the expansion

$$ve^{i\,\theta} - \frac{1}{v}e^{-i\,\theta} = ve^{i\,\theta_c} - \frac{1}{v}e^{-i\,\theta_c} + i\left(ve^{i\,\theta_c} + \frac{1}{v}e^{-i\,\theta_c}\right)(\theta-\theta_c)$$

imply the Lemma. □

Let $\mathbf{E}_0\,\xi = \mathbf{E}\left(\xi\,\mathbf{I}_{\{y\in\Omega_0\}}\right)$, $\mathbf{E}_1\,\xi = \mathbf{E}\left(\xi\,\mathbf{I}_{\{y\notin\Omega_0\}}\right)$.

Lemma 6.4. For any $s \geqslant 1$ there exists some positive constant $C(a,s)$ such that

$$\begin{aligned} \mathbf{E}_0 \Bigg| n \int_{\theta_c-\varepsilon}^{\theta_c+\varepsilon}\int_{t_c^+-\varepsilon}^{t_c^++\varepsilon} & \hat{g}_n(\theta,t)\, n^s\left(\Theta_n^\Gamma(u) + \Theta_n^\gamma(u)\right)^s\, \exp\left\{-n\,h^2(u)\frac{(t-t_c^+)^2}{2}\right\} \\ & \times \exp\left\{n\,h^2(u)\left(ve^{i\,\theta_c} - \frac{1}{v}e^{-i\,\theta_c}\right)^2\frac{(\theta-\theta_c)^2}{8}\right\}dt\,d\theta\Bigg| \leqslant \frac{C(a,s)}{n\,p^4(u)}. \end{aligned}$$

Proof. Let

$$\xi = \sqrt{n}\,h(u)\,(t-t_c^+)\,, \quad \zeta = -\frac{i}{2}\sqrt{n}\,h(u)\left(ve^{i\,\theta_c} - \frac{1}{v}e^{-i\,\theta_c}\right)(\theta-\theta_c)\,,$$

$$L^\Gamma = \sqrt{n}\,h(u)\,\rho\,, \quad L^\gamma = -\frac{i}{2}\sqrt{n}\,h(u)\left(ve^{i\,\theta_c} - \frac{1}{v}e^{-i\,\theta_c}\right)\rho\,, \quad -\varepsilon \leqslant \rho \leqslant \varepsilon\,.$$

We first consider the case $s \geqslant 2$. Noting that $|\hat{g}_n(\theta,t)| \leqslant C(a)$, for $|t-t_c^+| \leqslant \varepsilon$, $|\theta-\theta_c| \leqslant \varepsilon$ (see Lemma 6.3), and using Lemma 6.1, we get

$$\begin{aligned}
&\mathbf{E}_0 \left| n \int_{\theta_c-\varepsilon}^{\theta_c+\varepsilon} \int_{t_c^+-\varepsilon}^{t_c^++\varepsilon} \hat{g}_n(\theta,t)\, n^s \left(\Theta_n^\Gamma(u)+\Theta_n^\gamma(u)\right)^s \exp\left\{-n\,h^2(u)\frac{(t-t_c^+)^2}{2}\right\}\right.\\
&\qquad\qquad \left.\times \exp\left\{n\,h^2(u)\left(ve^{i\theta_c}-\frac{1}{v}e^{-i\theta_c}\right)^2\frac{(\theta-\theta_c)^2}{8}\right\} dt\,d\theta\right|\\
&= \frac{C(a)}{p(u)}\mathbf{E}_0 \left|\int_{L^\gamma} d\zeta \int_{L^\Gamma} d\xi\; \hat{g}_n(\zeta,\xi)\, n^s\left(\Theta_n^\Gamma(u)+\Theta_n^\gamma(u)\right)^s \exp\left\{-\frac{\xi^2}{2}\right\}\exp\left\{-\frac{\zeta^2}{2}\right\}\right|\\
&\leqslant \frac{C(a,s)}{p(u)}\mathbf{E}_0 \int_{L^\gamma} |d\zeta| \int_{L^\Gamma} |d\xi| \Bigg(\frac{|f_n'(z_c^+)-f'(z_c^+)|^s}{n^{-\frac{s}{2}}p^{\frac{s}{2}}(u)}(|\xi|^s+|\zeta|^s)\\
&+\frac{|f_n''(z_c^+)-f''(z_c^+)|^s}{p^s(u)}(|\xi|^{2s}+|\zeta|^{2s})+\frac{|\xi|^{3s}+|\zeta|^{3s}}{n^{\frac{s}{2}}p^{\frac{3}{2}s}(u)}\Bigg)\exp\left\{-\mathrm{Re}\,\frac{\xi^2}{2}\right\}\exp\left\{-\mathrm{Re}\,\frac{\zeta^2}{2}\right\}\\
&\leqslant \frac{C(a,s)}{p(u)}\int_{L^\gamma}|d\zeta|\int_{L^\Gamma}|d\xi|\left(\frac{|\xi|^s+|\zeta|^s+|\xi|^{2s}+|\zeta|^{2s}+|\xi|^{3s}+|\zeta|^{3s}}{n^{\frac{s}{2}}p^{\frac{3}{2}s}(u)}\right)\\
&\qquad\qquad \times\exp\left\{-\mathrm{Re}\,\frac{\xi^2}{2}\right\}\exp\left\{-\mathrm{Re}\,\frac{\zeta^2}{2}\right\} \leqslant \frac{C(a,s)}{n^{\frac{s}{2}}p^{\frac{3}{2}s+1}(u)}.
\end{aligned}$$

This completes the proof of the Lemma for $s \geqslant 2$. We now turn to the case $s=1$. Using Lemmas 6.1, 6.3, we obtain

$$\begin{aligned}
&\mathbf{E}_0 \left| n^2 \int_{\theta_c-\varepsilon}^{\theta_c+\varepsilon} \int_{t_c^+-\varepsilon}^{t_c^++\varepsilon} \hat{g}_n(\theta,t) \left(\Theta_n^\Gamma(u)+\Theta_n^\gamma(u)\right) \exp\left\{-n\,h^2(u)\frac{(t-t_c^+)^2}{2}\right\}\right.\\
&\qquad\qquad \left.\times \exp\left\{n\,h^2(u)\left(ve^{i\theta_c}-\frac{1}{v}e^{-i\theta_c}\right)^2\frac{(\theta-\theta_c)^2}{8}\right\} dt\,d\theta\right|\\
&= \frac{n\,C(a)}{p(u)}\mathbf{E}_0 \left|\int_{L^\gamma} d\zeta \int_{L^\Gamma} d\xi\; \hat{g}_n(\zeta,\xi)\left(\Theta_n^\Gamma(u)+\Theta_n^\gamma(u)\right) \exp\left\{-\frac{\xi^2}{2}\right\}\exp\left\{-\frac{\zeta^2}{2}\right\}\right|\\
&\leqslant \frac{n\,C(a)}{p(u)}\mathbf{E}_0 \int_{L^\gamma} |d\zeta| \int_{L^\Gamma} |d\xi| \Bigg(\frac{|f_n'(z_c^+)-f'(z_c^+)|+|f_n''(z_c^+)-f''(z_c^+)|}{n\,p(u)}\left(|\xi|^2+|\zeta|^2\right)\\
&\qquad\qquad +\frac{(|\xi|+|\zeta|)^6}{n^2p^3(u)}\Bigg)\exp\left\{-\mathrm{Re}\,\frac{\xi^2}{2}\right\}\exp\left\{-\mathrm{Re}\,\frac{\zeta^2}{2}\right\} \leqslant \frac{C(a)}{n\,p^4(u)}.
\end{aligned}$$

The Lemma follows. □

Lemma 6.5. There exist some positive constant $C(a)$ and a function

$|\phi(a,t,\theta)| \leqslant 1$ such that

$$\operatorname{Re}\Bigg[-\frac{n}{8\pi^2}\mathbf{E}_0\int_{\theta_c-\varepsilon}^{\theta_c+\varepsilon}\int_{t_c^+-\varepsilon}^{t_c^++\varepsilon}\hat{g}_n(\theta,t)\ \exp\left\{-n\,h^2(u)\frac{(t-t_c^+)^2}{2}\right\} \times\exp\left\{n\,h^2(u)\left(ve^{i\,\theta_c}-\frac{1}{v}e^{-i\,\theta_c}\right)^2\frac{(\theta-\theta_c)^2}{8}\right\}dt\,d\theta\Bigg] = \frac{p(u)}{\pi(1+4a^2)}+\phi(a,t,\theta)\frac{C(a)}{n\,p^4(u)}.$$

Proof. Let ξ, ζ, L^Γ, L^γ are the same as in Lemma 6.4, ψ is a complex function such that $|\psi| \leqslant 1$. Using Lemmas 6.1, 6.3, we get

$$\begin{aligned}
&-\frac{n}{8\pi^2}\mathbf{E}_0\int_{\theta_c-\varepsilon}^{\theta_c+\varepsilon}\int_{t_c^+-\varepsilon}^{t_c^++\varepsilon}\hat{g}_n(\theta,t)\ \exp\left\{-n\,h^2(u)\frac{(t-t_c^+)^2}{2}\right\}\\
&\qquad\times\exp\left\{n\,h^2(u)\left(ve^{i\,\theta_c}-\frac{1}{v}e^{-i\,\theta_c}\right)^2\frac{(\theta-\theta_c)^2}{8}\right\}dt\,d\theta\\
&=-\frac{n}{8\pi^2a^2}z_c^+h^2(u)\left(ve^{i\,\theta_c}-\frac{1}{v}e^{-i\,\theta_c}\right)\int_{\theta_c-\varepsilon}^{\theta_c+\varepsilon}\int_{t_c^+-\varepsilon}^{t_c^++\varepsilon}\exp\left\{-n\,h^2(u)\frac{(t-t_c^+)^2}{2}\right\}\\
&\qquad\times\exp\left\{n\,h^2(u)\left(ve^{i\,\theta_c}-\frac{1}{v}e^{-i\,\theta_c}\right)^2\frac{(\theta-\theta_c)^2}{8}\right\}dt\,d\theta\\
&+\psi\,n\,C(a)\mathbf{E}_0\Bigg|\int_{\theta_c-\varepsilon}^{\theta_c+\varepsilon}\int_{t_c^+-\varepsilon}^{t_c^++\varepsilon}\Big(\left|t-t_c^+\right|^2+|\theta-\theta_c|^2+\left|t-t_c^+\right||\theta-\theta_c|\\
&\quad+|f_n'(u\varkappa+i\,t)-f'(u\varkappa+i\,t)|+|f_n''(u\varkappa+i\,t)-f''(u\varkappa+i\,t)|\\
&\quad+|f_n'''(u\varkappa+i\,t)-f'''(u\varkappa+i\,t)|\,p(u)+|f_n''''(u\varkappa+i\,t)-f''''(u\varkappa+i\,t)|\,p^2(u)\Big)\\
&\quad\times\exp\left\{-n\,h^2(u)\frac{(t-t_c^+)^2}{2}\right\}\exp\left\{n\,h^2(u)\left(ve^{i\,\theta_c}-\frac{1}{v}e^{-i\,\theta_c}\right)^2\frac{(\theta-\theta_c)^2}{8}\right\}dt\,d\theta\Bigg|\\
&=\frac{z_c^+}{i\,4\pi^2a^2}\int_{L^\gamma}\exp\left\{-\frac{\zeta^2}{2}\right\}d\zeta\int_{L^\Gamma}\exp\left\{-\frac{\xi^2}{2}\right\}d\xi\\
&\qquad+\psi\frac{C(a)}{n\,p^3(u)}\int_{L^\gamma}d\zeta\int_{L^\Gamma}d\xi\,\left((|\xi|+|\zeta|)^2+1\right)\ \exp\left\{-\operatorname{Re}\frac{\xi^2}{2}\right\}\ \exp\left\{-\operatorname{Re}\frac{\zeta^2}{2}\right\}\\
&=\frac{p(u)}{\pi(1+4a^2)}-i\frac{u}{2\pi a^2}\frac{1+2a^2}{1+4a^2}+\psi\frac{C(a)}{n\,p^3(u)}.
\end{aligned}$$

This concludes the proof. □

Combining lemmas 6.2, 6.4, and 6.5, we obtain

$$\operatorname{Re}\left[\frac{n}{4\pi^2}\mathbf{E}_0\int_{\gamma_3^+}dz\int_{\Gamma_3^+}g_n(z,w)\,e^{n\{f_n(w)-f_n(z)\}}dw\right]=\frac{p(u)}{\pi(1+4a^2)}+\phi(a,t,\theta)\frac{C(a)}{n\,p^4(u)}. \quad (27)$$

Since $\overline{f_n(z)} = f_n(\overline{z})$, $\overline{g_n(z,w)} = g_n(\overline{z},\overline{w})$, we have

$$\operatorname{Re}\left[\frac{n}{4\pi^2}\mathbf{E}_0\int_{\gamma_3^-} dz \int_{\Gamma_3^-} g_n(z,w)\, e^{n\{f_n(w)-f_n(z)\}}dw\right] = \frac{p(u)}{\pi(1+4a^2)} + \phi(a,t,\theta)\frac{C(a)}{n\,p^4(u)}. \quad (28)$$

Noting that, for $z \in \gamma_3^b$, $w \in \Gamma_3^d$, $b, d \in \{+,-\}$, $b \neq d$,

$$\begin{aligned} g_n(z,w) &= \frac{z_c^b}{z_c^b - z_c^d} f''(z_c^b)(z - z_c^b) - \frac{z_c^d}{z_c^b - z_c^d} f''(z_c^d)(w - z_c^d) \\ &+ \psi(a,z,w)\frac{C(a)}{p(u)}\Big(|f_n'(z) - f'(z)| + |f_n'(w) - f'(w)| \\ &+ \left(|z - z_c^b| + |w - z_c^d|\right)^2 + \left(|z - z_c^b| + |w - z_c^d|\right)^3 + \left(|z - z_c^b| + |w - z_c^d|\right)^4 \Big), \end{aligned}$$

where $|\psi(a,z,w)| \leqslant 1$, we get

$$\begin{aligned} n\,\mathbf{E}_0 \Bigg| \int_{b\theta_c-\varepsilon}^{b\theta_c+\varepsilon} \int_{t_c^d-\varepsilon}^{t_c^d+\varepsilon} &\hat{g}_n(\theta,t)\ \exp\left\{-n\, f''(z_c^d)\frac{(t-t_c^d)^2}{2}\right\} \\ &\times \exp\left\{n\, f''(z_c^b)\left(ve^{i\,b\theta_c} - \frac{1}{v}e^{-i\,b\theta_c}\right)^2 \frac{(\theta - b\theta_c)^2}{8}\right\} dt\, d\theta \Bigg| \leqslant \frac{C(a)}{n\,p^4(u)}. \end{aligned}$$

Taking into account Lemma 6.4, we obtain

$$\operatorname{Re}\left[\frac{n}{4\pi^2}\mathbf{E}_0\int_{\gamma_3^b} dz \int_{\Gamma_3^d} g_n(z,w)\, e^{n\{f_n(w)-f_n(z)\}}dw\right] = \phi(a,t,\theta)\frac{C(a)}{n\,p^4(u)}, \quad (29)$$

for $b \neq d$. Now, using Theorem 4.1, inequalities (27), (28), and (29), we can conclude that

$$\mathbf{E}_0\, p_n^a(u,y) = \frac{2\,p(u)}{\pi(1+4a^2)} + \phi(a,t,\theta)\frac{C(a)}{n\,p^4(u)}.$$

From here we have

$$p_n^a(u) = \mathbf{E}_0\, p_n^a(u,y) + \mathbf{E}_1\, p_n^a(u,y) = \frac{2\,p(u)}{\pi(1+4a^2)} + \phi(a,t,\theta)\frac{C(a)}{n\,p^4(u)} + \mathbf{E}_1\, p_n^a(u,y),$$

where $1 + 4a^2 - u^2 \geqslant c\,n^{-\frac{1}{3}+\nu}$.
This completes the proof of Theorem 1.1.
From this theorem it is easy to get

Proposition 6.6. Let $\nu > 0$, $c > 0$. There exists a positive constant $C(a)$ such that for any $x \in [-\sqrt{1+4a^2} + c\,n^{-\frac{1}{3}+\nu}, \sqrt{1+4a^2} - c\,n^{-\frac{1}{3}+\nu}]$ we have

$$|\mathbf{E}\, F_n^a(x) - G^a(x)| \leqslant \frac{C(a)}{n(1+4a^2-x^2)}.$$

Proof. Since $\mathbf{E}\, F_n^a(0) = G^a(0)$, we get

$$\begin{aligned}|\mathbf{E}\, F_n^a(x) - G^a(x)| &\leqslant \int_0^{|x|} |p_n^a(u) - g^a(u)|\, du \\ &\leqslant \frac{C(a)}{n}\int_0^{|x|} \frac{du}{(1+4a^2-u^2)^2} + \int_0^{|x|} \mathbf{E}_1\, p_n^a(u,y)du \\ &\leqslant \frac{C(a)}{n(1+4a^2-x^2)} + \mathbf{E}_1 \int_0^{|x|} p_n^a(u,y)du \leqslant \frac{C(a)}{n(1+4a^2-x^2)}.\end{aligned}$$

The last inequality follows from (11). □

7 The proof of Theorem 1.2

This theorem is not a simple corollary of the local limit Theorem 1.1 and we need in some additional arguments. The proof is identical with the proof of Theorem 1.2 in [12]. For readers convenience we include some of these arguments.

Let $s_n^a(z)$ be the Stieltjes transform of the expected spectral distribution function $\mathbf{E}\, F_n^a(x)$ and $s^a(z)$ be the Stieltjes transform of the semi-circle distribution function $G^a(x)$, $z = u + i\, v$, that is

$$s^a(z) = -\frac{2}{1+4a^2}\left(z - \sqrt{z^2 - (1+4a^2)}\right), \qquad s_n^a(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} d\mathbf{E}\, F_n^a(x).$$

Consider the following intervals

$$\begin{aligned}\mathbf{J}_\varepsilon^{(l)} &= [-\sqrt{1+4a^2}+\varepsilon, -\sqrt{1+4a^2}+\eta], &\quad \mathbf{J}_\varepsilon^{(r)} &= [\sqrt{1+4a^2}-\eta, \sqrt{1+4a^2}-\varepsilon], \\ \mathbf{J}_\varepsilon^{\prime(l)} &= [-\sqrt{1+4a^2}, -\sqrt{1+4a^2}+\eta+\varepsilon], &\quad \mathbf{J}_\varepsilon^{\prime(r)} &= [\sqrt{1+4a^2}-\eta-\varepsilon, \sqrt{1+4a^2}].\end{aligned}$$

In what follows we shall assume that $\eta = C\, n^{-\frac{1}{3}+\nu}$, $\varepsilon = C\, n^{-\frac{1}{2}}$.

Lemma 7.1. Let $v > 0$, d and η be positive numbers such that

$$\gamma = \frac{1}{\pi}\int_{|u|\leqslant d} \frac{1}{u^2+1} du > \frac{1}{2},$$

and

$$\eta \geqslant 2\varepsilon \geqslant 2vd.$$

Then there exist some positive constants $C_1(\gamma), C_2(\gamma), C_3(\gamma), C_4(\gamma)$, depending on γ such that

$$\begin{aligned}\Delta &:= \sup_x |\mathbf{E}\, F_n^a(x) - G^a(x)| \\ &\leqslant C_1(\gamma) \sup_{x\in \mathbf{J}_\varepsilon^{\prime(l)}\cup \mathbf{J}_\varepsilon^{\prime(r)}} \left|\mathrm{Im}\left(\int_{-\infty}^{x} (s_n^a(z) - s^a(z))du\right)\right| + C_2(\gamma)\, v\sqrt{\eta} + C_3(\gamma)\, \varepsilon^{\frac{3}{2}} + \frac{C_4(\gamma)}{n\eta}.\end{aligned}$$

Proof. Introduce the following notation

$$\Delta_\varepsilon := \sup_{x\in \mathbf{J}_\varepsilon^{(l)}\cup \mathbf{J}_\varepsilon^{(r)}} |\mathbf{E}\, F_n^a(x) - G^a(x)|.$$

Proposition 6.6 now implies the inequality

$$\Delta \leqslant \Delta_\varepsilon + C_1\varepsilon^{\frac{3}{2}} + \frac{C_2}{n\eta}. \tag{30}$$

We see that

$$\begin{aligned}
\frac{1}{\pi}\mathrm{Im}\left(\int_{-\infty}^{x}(s_n^a(z)-s^a(z))du\right) &= \frac{1}{\pi}\int_{-\infty}^{x}\left[\int_{-\infty}^{\infty}\frac{v\, d\,(\mathbf{E}\, F_n^a(s)-G^a(s))}{(s-u)^2+v^2}\right]du\\
&= \frac{1}{\pi}\int_{-\infty}^{x}\left[\int_{-\infty}^{\infty}\frac{2(v(s-u)(\mathbf{E}\, F_n^a(s)-G^a(s))ds}{((s-u)^2+v^2)^2}\right]du\\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}(\mathbf{E}\, F_n^a(s)-G^a(s))\left[\int_{-\infty}^{x}\frac{2v(s-u)du}{((s-u)^2+v^2)^2}\right]ds\\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(\mathbf{E}\, F_n^a(x-vs)-G^a(x-vs))\, ds}{s^2+1}.
\end{aligned}$$

Furthermore, since $F_n^a(x)$ is non decreasing, we obtain

$$\begin{aligned}
\frac{1}{\pi}\int_{|s|<d}\frac{\Big(\mathbf{E}\, F_n^a(x-vs)-G^a(x-vs)\Big)ds}{s^2+1} &\geqslant \gamma\Big(\mathbf{E}\, F_n^a(x-vd)-G^a(x-vd)\Big)\\
&\quad - \frac{1}{\pi}\int_{|s|<d}\Big|G^a(x-vs)-G^a(x-vd)\Big|ds\\
&\geqslant \gamma\Big(G^a(x-vd)-G^a(x-vd)\Big)\\
&\quad - \frac{1}{v\pi}\int_{|s|<vd}\Big|G^a(x-s)-G^a(x-vd)\Big|ds.
\end{aligned}$$

Let $x_n \in \mathbf{J}_\varepsilon^{(l)}\cup\mathbf{J}_\varepsilon^{(r)}$ such that $\mathbf{E}\, F_n^a(x_n)-G^a(x_n)\to\Delta_\varepsilon$. Then $x_n \pm vd \in \mathbf{J}_\varepsilon^{\prime(l)}\cup\mathbf{J}_\varepsilon^{\prime(r)}$ and we have

$$\begin{aligned}
\sup_{x\in\mathbf{J}_\varepsilon^{\prime(a)}\cup\mathbf{J}_\varepsilon^{\prime(b)}}\left|\frac{1}{\pi}\mathrm{Im}\left(\int_{-\infty}^{x}(s_n^a(z)-s^a(z))du\right)\right| &\geqslant \gamma\lim_{n\to\infty}\Big(\mathbf{E}\, F_n^a(x_n)-G^a(x_n)\Big)\\
&- \frac{1}{v\pi}\sup_{x\in\mathbf{J}_\varepsilon^{\prime(l)}\cup\mathbf{J}_\varepsilon^{\prime(r)}}\int_{|s|<2vd}|G^a(x+s)-G^a(x)|ds-(1-\gamma)\Delta\\
&\geqslant \gamma\Delta_\varepsilon - C_1 d\, v\sqrt{\eta}-(1-\gamma)\Delta \geq (2\gamma-1)\Delta - C_1 d\, v\sqrt{\eta} - C_2\varepsilon^{\frac{3}{2}} - \frac{C_3}{n\eta}.
\end{aligned}$$

In the last inequality we use (30). Similar arguments may be used for the sequence $x_n \in \mathbf{J}_\varepsilon^{(l)}\cup\mathbf{J}_\varepsilon^{(r)}$ such that $\mathbf{E}\, F_n^a(x_n)-G^a(x_n)\to-\Delta_\varepsilon$. The Lemma follows. ☐

Lemma 7.2. Under the conditions of Lemma 7.1, for any $V > v$, the following inequality holds

$$\sup_{x\in \mathbf{J}'^{(l)}_\varepsilon \cup \mathbf{J}'^{(r)}_\varepsilon} \Big| \int_{-\infty}^{x} \operatorname{Im}\,(s_n^a(z) - s^a(z))du \Big| \leqslant \int_{-\infty}^{\infty} \big|(s_n^a(u+iV) - s^a(u+iV))\big| du$$
$$+ \sup_{x\in \mathbf{J}'^{(l)}_\varepsilon \cup \mathbf{J}'^{(r)}_\varepsilon} \left| \operatorname{Im} \left\{ \int_{v_0}^{V} (s_n^a(x+iv) - s^a(x+iv))dv \right\} \right|.$$

For the proof of this lemma see [11], Lemma 2.2.

Combining Lemmas 7.1, 7.2, we get

Corollary 7.3. Under the conditions of Lemma 7.2 the following inequality holds

$$\Delta \leqslant C_1 \Bigg(\int_{-\infty}^{\infty} |s_n^a(u+iV) - s^a(u+iV)| du$$
$$+ \sup_{x\in \mathbf{J}'^{(l)}_\varepsilon \cup \mathbf{J}'^{(r)}_\varepsilon} \left| \operatorname{Im} \left\{ \int_{v_0}^{V} (s_n^a(x+iu) - s^a(x+iu))du \right\} \right| \Bigg) + C_2\, v_0 \sqrt{\eta} + C_3\, \varepsilon^{\frac{3}{2}} + \frac{C_4}{n\eta}.$$

Let $\mathbf{M}(k)$ be the matrix obtained from $\mathbf{M}$ by deleting the kth row and kth column, and let $\alpha'(k) = \big(M_{1k}, \dots, M_{(k-1)k}, M_{(k+1)k}, \dots, M_{nk}\big)$. Set

$$\varepsilon_k^a = \frac{1}{\sqrt{n}} M_{kk} - \frac{1}{n} \overline{\alpha}'(k) \Big(\mathbf{M}(k) - z\mathbf{I}_{n-1} \Big)^{-1} \alpha(k) + \frac{1+4a^2}{4} s_n^a(z),$$

where $\mathbf{I}_{n-1}$ denotes the $(n-1)\times(n-1)$ identity matrix. Introduce

$$\delta_n^a(z) = -\frac{1}{n} \sum_{k=1}^{n} \mathbf{E}\, \varepsilon_k^a \frac{1}{(z + \frac{1+4a^2}{4} s_n^a(z))(z + \frac{1+4a^2}{4} s_n^a(z) - \varepsilon_k^a)}.$$

Then it is easy to see that

$$s_n^a(z) = -\frac{4}{4z + (1+4a^2)s_n^a(z)} + \delta_n^a(z). \tag{31}$$

In [11] it was proved that there exists some positive constant C_0 such that, for any $|u| \leqslant \sqrt{1+4a^2}$, $v \geqslant C_0\, n^{-\frac{1}{2}}$,

$$|\delta_n^a(z)| \leqslant \frac{C}{nv}. \tag{32}$$

Solving the equality (31) and choosing a solution with $\operatorname{Im} s_n^a(z) > 0$, we obtain

$$s_n^a(z) = -\frac{2}{1+4a^2} \left(z - \frac{1+4a^2}{4} \delta_n^a(z) - \sqrt{\left(z + \frac{1+4a^2}{4} \delta_n^a(z) \right)^2 - (1+4a^2)} \right),$$

for $|u| \leqslant \sqrt{1+4a^2}$, $v \geqslant C_0\, n^{-\frac{1}{2}}$. From the last equality it follows that

$$s_n^a(z) = \delta_n^a(z) + s^a\left(z + \frac{1+4a^2}{4} \delta_n^a(z) \right),$$

$$|s_n^a(z) - s^a(z)| \leqslant |\delta_n^a(z)| + \left| s^a(z) - s^a\left(z + \frac{1+4a^2}{4}\delta_n^a(z)\right)\right|.$$

After a simple calculation we get

$$|s_n^a(z) - s^a(z)| \leqslant |\delta_n^a(z)| \left(\frac{3}{2} + \frac{C(a)}{\left|\sqrt{z^2-(1+4a^2)} - \sqrt{\left(z+\frac{1+4a^2}{4}\delta_n^a(z)\right)^2 - (1+4a^2)}\right|} \right).$$

Since $\operatorname{Re}\sqrt{z^2-(1+4a^2)}\operatorname{Re}\sqrt{\left(z+\frac{1+4a^2}{4}\delta_n^a(z)\right)^2 - (1+4a^2)} \geq 0$, we have, for $z = u + iv$ such that $u \in \mathbf{J'}_\varepsilon^{(l)} \cup \mathbf{J'}_\varepsilon^{(r)}$, and $v \geqslant C_0 n^{-\frac{1}{2}}$,

$$|s_n^a(z) - s^a(z)| \leqslant |\delta_n^a(z)| \left(\frac{3}{2} + \frac{C(a)}{\left|\sqrt{z^2-(1+4a^2)}\right|} \right) \leqslant |\delta_n^a(z)| \left(\frac{3}{2} + \frac{C(a)}{\sqrt{\varepsilon}} \right).$$

From here, using (32), we obtain

$$|s_n^a(z) - s^a(z)| \leqslant \frac{C(a)}{n\,v}(1+\varepsilon^{-\frac{1}{2}}),$$

for $u \in \mathbf{J'}_\varepsilon^{(l)} \cup \mathbf{J'}_\varepsilon^{(r)}$, and $v \geqslant C_0 n^{-\frac{1}{2}}$. Choosing $V = 1$ in Corollary 7.3 and taking into account that $\int_{-\infty}^{\infty} |s_n^a(u+i) - s^a(u+i)| du \leqslant \frac{C(a)}{n}$ (see [11], (4.29)), after integration in u and v we get

$$\Delta \leqslant \frac{C_1}{n} + C_2\frac{|\ln v_0|}{n}(1+\varepsilon^{-\frac{1}{2}}) + C_3\, v_0\sqrt{\eta} + C_4\,\varepsilon^{\frac{3}{2}} + \frac{C_5}{n\eta}.$$

Using the last inequality with $\eta = C\,n^{-\frac{1}{3}+\nu}$, $\varepsilon = C\,n^{-\frac{1}{2}}$, $v_0 = C\,n^{-\frac{1}{2}}$, we get

$$\Delta \leqslant C(a)\,n^{-\frac{2}{3}+\nu}.$$

Theorem 1.2 is proved.

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